MATHEMATICS-II

Garima Singh

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FOREWORD

Engineering has played a very significant role in the progress and expansion of mankind and society for centuries. Engineering ideas that originated in the Indian subcontinent have had a thoughtful impact on the world.

All India Council for Technical Education (AICTE) had always been at the forefront of assisting Technical students in every possible manner since its inception in 1987. The goal of AICTE has been to promote quality Technical Education and thereby take the industry to a greater heights and ultimately turn our dear motherland India into a Modern Developed Nation. It will not be inept to mention here that Engineers are the backbone of the modern society - better the engineers, better the industry, and better the industry, better the country.

NEP 2020 envisages education in regional languages to all, thereby ensuring that each and every student becomes capable and competent enough and is in a position to contribute towards the national growth and development.

One of the spheres where AICTE had been relentlessly working from last few years was to provide high-quality moderately priced books of International standard prepared in various regional languages to all it's Engineering students. These books are not only prepared keeping in mind it's easy language, real life examples, rich contents and but also the industry needs in this everyday changing world. These books are as per AICTE Model Curriculum of Engineering & Technology – 2018.

Eminent Professors from all over India with great knowledge and experience have written these books for the benefit of academic fraternity. AICTE is confident that these books with their rich contents will help technical students master the subjects with greater ease and quality.

AICTE appreciates the hard work of the original authors, coordinators and the translators for their endeavour in making these Engineering subjects more lucid.

(Anil D. Sahasrabudhe)

Acknowledgement

The author is grateful to AICTE for their meticulous planning and execution to publish the technical book for Diploma students.

I sincerely acknowledge the valuable contributions of the reviewer of the book Prof. Billu Ram Saini, for making it students' friendly and giving a better shape in an artistic manner.

This book is an outcome of various suggestions of AICTE members, experts and authors who shared their opinion and thoughts to further develop the engineering education in our country.

It is also with great honour that I state that this book is aligned to the AICTE Model Curriculum and in line with the guidelines of National Education Policy (NEP) -2020. Towards promoting education in regional languages, this book is being translated in scheduled Indian regional languages.

Acknowledgements are due to the contributors and different workers in this field whose published books, review articles, papers, photographs, footnotes, references and other valuable information enriched us at the time of writing the book.

Finally, I like to express my sincere thanks to the publishing house, M/s. Khanna Book Publishing Company Private Limited, New Delhi, whose entire team was always ready to cooperate on all the aspects of publishing to make it a wonderful experience.

Garima Singh

Preface

Mathematics is inextricably woven to all the technological aspects of human kind. An in-depth knowledge of mathematics is of paramount importance when a student enters the world of technology. When applied to technology, it allows scientists and engineers to produce systematic, reproducible, and transmittable knowledge.

The book "Mathematics-II" is primarily designed for the students of diploma engineering(common to all branches) to tackle the 21st century and onward technological challenges. It is strictly aligned to the AICTE's model curriculum for diploma courses in engineering and technology, incorporating student's oriented and self-learning activities as per New National Education Policy 2020. Outcome Based Education and Bloom's Taxonomy concepts are the central ideas behind the book's layout. Each topic in the book has been treated in a lucid and easy style so as to make the mathematical language simple and crisp. There has been a deliberate attempt to keep the number of pages in the book minimum without compromising with the matter. While preparing the manuscript, various standard textbooks, reference books (a few mentioned in the reference section too) has been referred and accordingly the sections have been developed. Efforts have been made to explain the fundamental concepts of the subject in the simplest possible way so as to make learning a pleasure.

This book comprises of five units. There is a uniformity maintained in writing all the units. Each unit starts with the unit specifics, rationale and pre-requisites. Apart from the theory explanation and solved examples, mini-projects, activity, fun-facts, QR codes, case studies, video resources, real life applications have been incorporated so as to enhance interactive understanding and student's applicability skills,which make them competitive and employable. Check-out section has been introduced so as to activate the curiosity part of the student by corelating all the topics studied in this book with MATLAB. The text has been supplemented with notes, remarks, remember sections within grey boxes. In addition, some useful information has been given under the heading 'Know More'. Relevant essential basic information has been incorporated in the Appendices. An attempt has been made to enrich the book by including a few activities in the Annexures part. Overall, an approach has been tried to made so as to discourage rote memorization. For direct recapitulation of main concepts, formulae and results, brief summary of the unit has been given.

At the end of each unit ,an excerpt related to eminent Indian Mathematicians is given so as to make students have a glimpse of the rich Indian heritage, especially in the field of mathematics.

I sincerely hope that the book will motivate and inspire students to learn and apply basics of mathematics and will definitely contribute towards solid foundation building of the subject. I would be grateful to acknowledge any comments/suggestions from the teachers/students/readers towards further improvement of the book in future editions. It was indeed a pleasure writing the book covering varied topics in a crisp manner for future leaders to make fundamental contributions towards society.

Garima Singh

Outcome Based Education

For the implementation of an outcome based education the first requirement is to develop an outcome based curriculum and incorporate an outcome based assessment in the education system. By going through outcome based assessments, evaluators will be able to evaluate whether the students have achieved the outlined standard, specific and measurable outcomes. With the proper incorporation of outcome based education there will be a definite commitment to achieve a minimum standard for all learners without giving up at any level. At the end of the programme running with the aid of outcome based education, a student will be able to arrive at the following outcomes:

Programme Outcomes (POs) are statements that describe what students are expected to know and be able to do upon graduating from the program. These relate to the skills, knowledge, analytical ability attitude and behaviour that students acquire through the program. The POs essentially indicate what the students can do from subject-wise knowledge acquired by them during the program. As such, POs define the professional profile of an engineering diploma graduate.

National Board of Accreditation (NBA) has defined the following seven POs for an Engineering diploma graduate:

PO1. Basic and Discipline specific knowledge: Apply knowledge of basic mathematics, science and engineering fundamentals and engineering specialization to solve the engineering problems.

PO2. Problem analysis: Identify and analyses well-defined engineering problems using codified standard methods.

PO3. Design/ development of solutions: Design solutions for well-defined technical problems and assist with the design of systems components or processes to meet specified needs.

PO4. Engineering Tools, Experimentation and Testing: Apply modern engineering tools and appropriate technique to conduct standard tests and measurements.

PO5. Engineering practices for society, sustainability and environment: Apply appropriate technology in context of society, sustainability, environment and ethical practices.

PO6. Project Management: Use engineering management principles individually, as a team member or a leader to manage projects and effectively communicate about well-defined engineering activities.

PO7. Life-long learning: Ability to analyse individual needs and engage in updating in the context of technological changes.

Course Outcomes

By the end of the course the students are expected to learn:

CO-1: The necessary background in matrices and determinants so as to apply them in finding solutions and aid in interpreting/analysing linear systems, optimization tactics.

CO-2: Determining the area and volume especially by applying simple techniques of Integral calculus.

CO-3: To analyse that coordinate geometry provides a connection between algebra and geometry through graphs of lines and curves.

CO-4: To tell the difference between a resultant and a concurrent force; to interpret and analyse simple physical problems in the form of a differential equation.

CO-5: To explore and visualize data by using the applicability of topics learnt and also with the help of some basics of MATLAB.

Mapping of Course Outcomes with Programme Outcomes to be done according to the matrix given below:

Course Outcomes	Expected Mapping with Programme Outcomes (1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)						
	PO-1	PO-2	PO-3	PO-4	PO-5	PO-6	PO-7
CO-1	3	3	3	3	1	1	3
CO-2	3	2	2	2	1	1	3
CO-3	3	2	2	2	1	1	3
CO-4	3	2	2	3	1	1	3
CO-5	3	3	3	3	1	1	3

Symbols and Abbreviations

Symbol/ Abbreviation	Name of Sign/Full form	Symbol/ Abbreviation	Name of Sign/Full form
[A:B] or [A/B]	Augmented matrix	$\frac{dy}{dx}$	Differential operator of variable y w.r.t. variable x
$\overline{AB}$	Line segment AB	$\int$	Integral
$\lvert AB\rvert$	The length of $\overline{AB}$	!	Factorial
$\overrightarrow{AB}$	Ray AB	$\in$	Is an element of/belongs to
CO	Course outcomes	$\notin$	Is not an element of/does not belong to
UO	Unit outcomes	$\neq$	Is not equal to
PO	Programme outcomes	$\sim$	Is similar to
N	Set of natural numbers	$\parallel$	Is parallel to
W	Set of whole numbers	$\approx$	Is approximately equal to
Z	Set of integers	( )	Parentheses (grouping symbol)
Q	Set of rational numbers	[ ]	Square brackets (grouping symbol)
R	Set of real numbers	{ }	Brace or curly brackets (grouping symbol)
C	Set of complex numbers	$\cong$	Is congruent to
I	Set of irrational numbers	$\sqrt[3]{\ }$	Cube root
Lf′ (a)	Left hand derivative of ‘f’ at ‘a’	Σ	The sum of
Rf′ (a)	Right hand derivative of ‘f’ at ‘a’	$\subset$ or $\subseteq$	Is a sub set of
L.H.S.	Left hand side	$\not\subset$ or $\not\subseteq$	Is not a subset
R.H.S.	Right hand side	$\cup$	Union of
adj (A)	Adjoint of matrix A	$\cap$	The intersection of
lim	Limit	ϕ	Empty set/null set
$f^n(a)$	n^{th} derivative of (f) at ‘a’	$\Rightarrow$	This implies
s.t.	Such that	$\Leftrightarrow$	Implies and is implied by
w.r.t.	With respect to	$\lvert\ \rvert$	Modulus
$\forall$	For all	$\therefore$	Therefore
$\lVert\ \rVert$	Norm	$\because$	Because

List of Figures

Unit 1: Determinants and Matrices

Unit 2: Integral Calculus

Unit 3: Co-ordinate Geometry

Unit 4: Vector Algebra

Unit 5: Differential Equations

The implementation of Outcome Based Education (OBE) framework and enhanced focus on the use of

Guidelines for Teachers

Bloom's Taxonomy necessitates that knowledge level and professional skills of the students should be enhanced. Teachers should take a major responsibility for the proper implementation of OBE. Some of the responsibilities (not limited to) for the teachers in OBE system may be as follows:

- Within reasonable constraint, they should manipulate time to the best advantage of all students.
- They should assess the students only upon certain defined criterion without considering any other potential ineligibility to discriminate them.
- They should try to grow the learning abilities of the students to a certain level before they leave the institute.
- They should try to ensure that all the students are equipped with the quality knowledge as well as competence after they finish their education.
- They should always encourage the students to develop their ultimate performance capabilities.
- They should facilitate and encourage group work and team work to consolidate newer approach.
- They should follow Blooms taxonomy in every part of the assessment.

Bloom's Taxonomy

Level	Teacher should Check	Student should be able to	Possible Mode of Assessment
Creating	Students ability to create	Design or Create	Mini project
Evaluating	Students ability to Justify/evaluate	Explain or Defend	Assignment
Analysing	Students ability to Scrutinize	Differentiate or examine	Project Methodology/Case Study
Applying	Students ability to use information	Solve or implement	Presentation/ Demonstration of solution of real life problems
Understanding	Students ability to explain the ideas	Explain or Classify	Presentation/Seminar
Remembering	Students ability to recall (or remember)	Define or Recall	Quiz

Guidelines for Students

Students should take equal responsibility for implementing the OBE. Some of the responsibilities (not limited to) for the students in OBE system are as follows:

- Students should be well aware of each UO before the start of a unit in each and every course.
- Students should be well aware of each CO before the start of the course.
- Students should be well aware of each PO before the start of the programme.
- Students should think critically and reasonably with proper reflection and action.
- Learning of the students should be connected and integrated with practical and real-life consequences.
- Students should be well aware of their competency at every level of OBE.

Contents

1 Determinants and Matrices

UNIT SPECIFICS

This unit incorporates the concepts of elementary properties of determinants up to 3rd order, consistency of equations, Cramer's rule, algebra of matrices, inverse of a matrix, matrix inverse method to solve a system of linear equations in 3 variables. Emphasis is on understanding the applicability with the help of solved and unsolved examples.

RATIONALE

Everything we learn has an application. And when it comes to matrices and determinants, they are omnipresent. They are everywhere around us. Matrices are a compact and easy way of articulating things, which can be visualized easily. Most of the computer simulations you see now a days use matrices. Nearly every branch of engineering, physics, economics, combinatorics, network analysis, operations research, epidemiology, communication and many others use matrices widely. Determinants give a definite value to these matrices and so are another face of the same coin.

Hence, when it comes to one's competency, especially technically, an in depth study of these cannot be ruled out. Even for higher studies these are pre-requisites for quite a many topics. Applicability comes when basics are mirror clear. In today's world when we are surrounded by numerous 'apps' for nearly each discipline, an understanding of matrices and determinants is must.

In short, as matrices and determinants connect to so many topics, they are extremely efficient tools in modelling things mathematically!

Pre-requisites

- Addition, subtraction, multiplication of numbers and polynomials.
- Basics of linear equations.
- Basic knowledge of trigonometric formulae.

Unit-I Determinants and matrices	UNIT OUTCOMES The student will learn -
UI-O1	To use properties of determinants in solving multifarious problems.
UI-O2	To correlate real life problems with system of linear equations and test/interpret, the consistency of equations especially with the help of Cramer's rule.
UI-O3	To perform common matrix operations like addition, scalar multiplication, multiplication, transposition.

UI-O4	To solve linear system of equations using the language of matrices.
UI-O5	To apply matrices and determinants in toto for comprehending problems.

CO-UO-Mapping

Unit-I Outcome	EXPECTED MAPPING WITH COURSE OUTCOMES (1 - Weak Correlation; 2 - Medium Correlation; 3 - Strong Correlation)				
	CO1	CO2	CO3	CO4	CO5
UI-O1	3	1		2	1
UI-O2	3		1		1
UI-O3	3		2		1
UI-O4	3		1		1
UI-O5	3		1		2

TOPIC 1 DETERMINANTS

1.1 INTRODUCTION

The concept of determinants evolved over a number of years. Gabriel Cramer added to the theory of determinants in relation to the sets of equation. Many other academicians like Arthur Cayley, C.G.J. Jacobi, J.J. Sylvester etc. enriched the theory of determinants.

Knowledge of determinants is imperative especially in the field of technical applications,though they have wide applications in umpty number of areas like science, social science ,economics etc. In this unit, we would study determinants up to third order.

Definition

D**eterminants** are defined as a scalar values associated with square matrices. So, to every square matrix* A = $[a_{ij}]$ of order n, we can associate a number (real or complex) called **determinant** of the square matrix A. It is denoted by det A or |A| or Δ.

Cramer's Rule was given by Swiss mathematician Gabriel Cramer (1704-1752) in 1750 in Introduction ál' Analyse des lignes Courbes algebriques.

Determinant of order 1

These are trivial determinants and the **value** of determinant is just the number in the matrix. They have one row and one column

For ex: If $|A| = |2|_{1\times1}$, then $|A| = 2$

Determinant of order 2

Consider the system of equation $a_1x + b_1 = 0$ and $a_2x + b_2 = 0$. If these are satisfied by the same value of x then, $a_1b_2 - a_2b_1 = 0$ and the expression $a_1b_2 - a_2b_1$ is called the **value** of the determinant of order 2 and is denoted by

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

* See page 14 for matrix.

The quantities a_1, a_2, b_1, b_2 are called the **elements or constituent's of the** determinant. A determinant of order 2 consists of two rows (horizontal elements) and two columns(vertical elements).

Next, consider the two determinants

$$\begin{vmatrix} 9 & 1 \\ 4 & 2 \end{vmatrix} = (9 \times 2) - (4 \times 1) = 18 - 4 = 14.$$

and $$\begin{vmatrix} 7 & 0 \\ 20 & 2 \end{vmatrix} = (7 \times 2) - (20 \times 0) = 14 - 0 = 14.$$

We see that $$\begin{vmatrix} 9 & 1 \\ 4 & 2 \end{vmatrix} = \begin{vmatrix} 7 & 0 \\ 20 & 2 \end{vmatrix}$$

Hence determinant is not simply a system of numbers. It has got a numerical value.

Note: Sign system for determinant of order 2 is $\begin{vmatrix} + & - \\ - & + \end{vmatrix}$

Example 1. Calculate the value of x if $\Delta = \begin{vmatrix} x & 3 \\ 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$

Solution: It is given that $\begin{vmatrix} x & 3 \\ 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$ which implies, $(x \times 2) - (4 \times 3) = (3 \times 2) - (1 \times 2)$

$\Rightarrow \quad 2x - 12 = 6 - 2 \quad \Rightarrow \quad 2x = 16 \quad \Rightarrow \quad x = 8$ **Ans.**

Determinants of order 3

Consider the system of equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$

If these equations are satisfied by the same values of x and y then on eliminating x and y, we get

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

The above expression on the left (L.H.S.) is called a determinant of the third order and is denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

FUN FACT
The term 'determinant' was first introduced by Carl Friedrich Gauss in his book 'Disquisitiones arithmeticae' in 1798 (published in 1801) at the age of 21!

A determinant of third order consists of three rows and three columns. The quantities a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 are called the elements or constituents of the determinant.

Note 1: Sign system for determinants of order three is

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$ (alternate '+' and '–', starting by taking principal element '+')

Note 2: Modulus function is totally different than determinants. For ex: in modulus $|-2| = 2$, whereas in determinants $|-2| = -2$.

1.1.1 Value/Expansion of Determinants of Third Order

(*a*) With respect to first row,

$$\begin{vmatrix} a_1 & \cdots & b_1 & \cdots & c_1 \\ a_2 & & b_2 & & c_2 \\ a_3 & & b_3 & & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

(*b*) With respect to second column,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ & \vdots & \\ a_2 & b_2 & c_2 \\ & \vdots & \\ a_3 & b_3 & c_3 \end{vmatrix} = -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= -b_1(a_2c_3 - a_3c_2) + b_2(a_1c_3 - a_3c_1) - b_3(a_1c_2 - a_2c_1)$$

Remark: A determinant can be expanded along any of its row or column. Value of the determinant remains same in any of the cases.

Rule of Sarrus: *It is a mnemonic device to get the value of determinant of order three* as follows

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{array}{c} \qquad\qquad\quad -ve \;\; -ve \;\; -ve \\ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} \\ \qquad\qquad\quad +ve \;\; +ve \;\; +ve \end{array}$$

$$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)$$

Note that the product of the terms in first bracket (*i.e.*, $a_1a_2a_3b_1b_2b_3c_1c_2c_3$) is same as the product of the term in second bracket.

Example 2. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix}$

Solution:

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 3 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= (3 + 3) - 2(-1 - 6) + 3(1 - 6) = 5$$

Alternative: By rule of Sarrus

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix}\begin{matrix} 1 & -2 \\ -1 & 3 \\ 2 & -1 \end{matrix}$$

$$= (3 + 12 + 3) - (18 - 3 - 2) = 18 - 13 = 5$$

1.1.2 Minors and Cofactors

Definition MINOR: The minor of a given element of determinant is the determinant obtained by deleting the row and column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ and the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

Hence a determinant of order three will have 9 minors.

Definition COFACTOR (C_{ij}): If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element is given by $C_{ij} = (-1)^{i+j} M_{ij}$

Example 3. Find the minors and cofactors of elements '–3', '5' and '–1' in the determinant $\begin{vmatrix} 1 & -3 & 2 \\ 3 & 0 & 5 \\ -1 & 1 & 7 \end{vmatrix}$

Solution:

$$\text{Minor of } -3 = \begin{vmatrix} 3 & 5 \\ -1 & 7 \end{vmatrix} = 26; \text{ Cofactor of } -3 = -26$$

$$\text{Minor of } 5 = \begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} = -2; \text{ Cofactor of } 5 = 2$$

$$\text{Minor of } -1 = \begin{vmatrix} -3 & 2 \\ 0 & 5 \end{vmatrix} = -15; \text{ Cofactor of } -1 = -15$$

1.1.3 Expansion of a Determinant in terms of Minors and Cofactors

(*i*) The sum of the product of elements of any row (column) with their corresponding cofactors is always equal to the value of the determinant.

(*ii*) The sum of the product of elements of any row (column) with the cofactors of other row (column) is always equal to zero.

1.1.4 Properties of Determinants

(*i*) The value of a determinant remains unaltered, If the rows and columns are inter–changed.

e.g., If
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(*ii*) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only *e.g.*,

Let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \Delta_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Then
$$\Delta_1 = -\Delta.$$

(*iii*) If all the elements of a row (or column) are zero, then the value of the determinant is zero.

(*iv*) If all the elements of any row (or column) are multiplied by the same number, then the determinant is multiplied by that number.

e.g., If $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Then $\Delta_1 = k\Delta$

(*v*) If all the elements of a row (or column) are proportional (or identical) to the element of any other row then the determinant vanishes, *i.e.*, its value is zero.

e.g., If $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \Delta = 0.$$

Also if $$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \Delta_1 = 0$$

(*vi*) If each element of any row (or column) is expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

e.g., $$\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(*vii*) **Row-column operation:** The value of a determinant remains unaltered under a column (C_i) operation of the form $C_i \to C_i + \alpha C_j + \beta C_k (j, k \neq i)$ or row (R_i) operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$ $(j, k \neq i)$. In other words, the value of a determinant is not altered by adding the elements of any row (or column) to the same multiples of the corresponding elements of any other row (or column)

e.g., Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then $$\Delta = \begin{vmatrix} a_1 + \alpha a_2 & b_1 + \alpha b_2 & c_1 + \alpha c_2 \\ a_2 & b_2 & c_2 \\ a_3 + \beta a_2 & b_3 + \beta b_2 & c_3 + \beta c_2 \end{vmatrix} (R_1 \to R_1 + \alpha R_2; R_3 \to R_3 + \beta R_2)$$

(*viii*) **Factor theorem:** If each element of a matrix A is a polynomial in x and if Δ vanishes for $x = a$, then $(x - a)$ is a **factor** of Δ.

Another form: If the elements of a determinant Δ are rational integral functions of x and two rows (or columns) become identical when $x = a$ then $(x - a)$ is a factor of Δ.

Note that if r rows become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of Δ.

Example 4. Find the value of determinant

$$\Delta = \begin{vmatrix} 2 & 1 & 3 \\ 5 & 0 & 2 \\ 2 & -1 & 7 \end{vmatrix}$$

Solution: Expanding along the second row, we get

$$\Delta = -5\begin{vmatrix} 1 & 3 \\ -1 & 7 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 2 & 7 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= -5(7+3) + 0 - 2(-2-2) = -50 + 8 = -42$$

Note: The element 5 occurs in the second row and the first column. We have $(-1)^{2+1} = (-1)^3 = -1$. Therefore we have fixed negative sign before 5. Then the sign before 0 will be positive and the sign before 2 will be negative.

1.1.5 Multiplication of Two Determinants

If $A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $B = \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}$.

Then, $$A \times B = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1l_1 + b_1l_2 & a_1m_1 + b_1m_2 \\ a_2l_1 + b_2l_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

Similarly, two determinants of order three are multiplied

(*a*) Here, we have multiplied row by column. We can also multiply row by row, column by row and column by column. (It should be noted that matrix multiplication and determinant multiplication have similar procedures.)

(*b*) If Δ_1 is the determinant formed by replacing the elements of determinant Δ of order n by their corresponding cofactors then $\Delta_1 = \Delta^{n-1}$

Example 5. Show that $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$

Solution: We have $$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{vmatrix}$$

(Applying row by column rule of multiplication.)

$$= \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

1.1.6 Nature of System of Linear Equations and Cramer's rule

I. Nature of System of Linear Equations with Two Variables

(*i*) Consistent equations

Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

(1) **Definite and unique solution [Intersecting lines]:** A System of (linear) equations is said to be consistent, if it has at least one unique solution.

Algebraically, $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

(2) **Infinite solutions [Identical lines]:** A system of (linear) equations is said to be consistent and dependent if it has infinite solutions.

Algebraically, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

$\Rightarrow$ Given equations are consistent and dependent.

(*ii*) Inconsistent equations: No solution [Parallel lines] - A system of (linear) equations is said to be inconsistent, if it has no solution.

Algebraically, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

$\Rightarrow$ Given equations are inconsistent.

II. Cramer's Rule for Solving System of Linear Equations

Cramer's rule is an explicit formula for the solution of a system of linear equations with as many equations as unknowns. It is applicable only if the value of corresponding determinant is non-zero. Here, we restrict our study to the equations with two and three variables.

System of Linear Equations (Cramer's Rule) in Two Variables

Let us consider a system of equations

$$\left.\begin{aligned} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{aligned}\right\} \quad \text{where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

On solving by cross- multiplication, we get

$$\frac{x}{(b_1c_2 - b_2c_1)} = \frac{y}{(c_1a_2 - c_2a_1)} = \frac{1}{(a_1b_2 - a_2b_1)} \quad \text{OR}$$

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{OR} \quad x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

System of Linear Equations in Three Variables (Cramer's Rule)

Let us consider a system of linear equations

$$a_1x + b_1y + c_1z = d_1 \qquad \ldots(i)$$

$$a_2x + b_2y + c_2z = d_2 \quad ...(ii)$$
$$a_3x + b_3y + c_3z = d_3 \quad ...(iii)$$

Here, $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

If $\Delta \neq 0$, then,

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta} \quad ...(iv)$$

The rule given in Eq. (*iv*) to find the values of x, y and z is called the **CRAMER's RULE.**

Remark:

1. Δ_i is obtained by replacing elements of i^{th} columns by d_1, d_2, d_3, where $i = 1,2,3$

III. Nature of Solutions of System of Linear Equations with Three Variables

Let us consider a system of linear equations be

$$a_1x + b_1y + c_1z = d_1 \quad ...(i)$$
$$a_2x + b_2y + c_2z = d_2 \quad ...(ii)$$
$$a_3x + b_3y + c_3z = d_3 \quad ...(iii)$$

Then, $$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta},$$

where Δ, Δ_1, Δ_2, Δ_3 haver their usual meaning

Note:

(*i*) If $\Delta \neq 0$ and atleast one of Δ_1, Δ_2, $\Delta_3 \neq 0$, then the given system of equations is consistent and has unique non trivial solution.

(*ii*) If $\Delta \neq 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then the given system of equations is consistent and has trivial solution only.

(*iii*) If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, then the given system of equations is consistent and has infinite solutions.

> Note that in case $\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_1x + b_1y + c_1z &= d_2 \\ a_1x + b_1y + c_1z &= d_3 \end{aligned}\right\}$ (Atleast two of d_1, d_2 and d_3 are not equal)
>
> $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$. But these three equations represent three parallel planes . Hence the system is inconsistent.

(*iv*) If $\Delta = 0$ but atleast one of Δ_1, Δ_2, Δ_3 is not zero then the equations are inconsistent and have no solution.

Example 6. Solve the following system of equations by Cramer's rule. Comment on their consistency.

$$x + y = 5 \quad \text{and} \quad 3x - 2y = 7$$

Solution: Here,

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2 - 3 = -5 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 5 & 1 \\ 7 & -2 \end{vmatrix} = -10 - 7 = -17 \quad \text{and} \quad \Delta_2 = \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = 7 - 15 = -8$$

Then, by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-17}{-5} = \frac{17}{5} \quad \text{and} \quad y = \frac{\Delta_2}{\Delta} = \frac{-8}{-5} = \frac{8}{5}$$

$$\therefore \qquad x = \frac{17}{5}, \quad y = \frac{8}{5}$$

So, the given system of linear equations are consistent as they have definite and unique solution (intersecting lines).

Example 7. Solve the following system of linear equations with the help of Cramer's rule. Comment on their consistency.

$$\begin{aligned} x + 2y + z &= 7 \\ 2x + 4y + 5z &= 8 \\ 3x + y + 9z &= 6 \end{aligned}$$

Solution: Let

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 1 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & -5 & 6 \end{vmatrix}, \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$= 15$$

Thus $\Delta \neq 0$ and therefore the system has a unique solution given by $\dfrac{x}{\Delta_1} = \dfrac{y}{\Delta_2} = \dfrac{z}{\Delta_3} = \dfrac{1}{\Delta}$, *i.e.*, given by

$$\frac{x}{\begin{vmatrix} 7 & 2 & 1 \\ 8 & 4 & 5 \\ 6 & 1 & 9 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 7 & 1 \\ 2 & 8 & 5 \\ 3 & 6 & 9 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 & 7 \\ 2 & 4 & 8 \\ 3 & 1 & 6 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 1 & 9 \end{vmatrix}}$$

Now

$$\Delta_1 = \begin{vmatrix} 7 & 2 & 1 \\ 8 & 4 & 5 \\ 6 & 1 & 9 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 1 \\ -6 & 0 & 3 \\ 5/2 & 0 & 17/2 \end{vmatrix} \text{ by } \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - \frac{1}{2}R_1 \end{matrix}$$

$$= -2\left[\left(-6 \times \frac{17}{2}\right) + \left(-3 \times \frac{5}{2}\right)\right]$$

$$= 102 + 15 = 117 \text{ (expanding along column 2)}$$

Again,

$$\Delta_2 = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 8 & 5 \\ 3 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 1 \\ 0 & -6 & 3 \\ 0 & -15 & 6 \end{vmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} = 39$$

Also $$\Delta_3 = \begin{vmatrix} 1 & 2 & 7 \\ 2 & 4 & 8 \\ 3 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 7 \\ 0 & 0 & -6 \\ 3 & 1 & 6 \end{vmatrix} R_2 \to R_2 - 2R_1$$

$$= -30 \text{ (expanding along row 2)}$$

$\therefore$ The solution is given by

$$\frac{x}{117} = \frac{y}{39} = \frac{z}{-30} = \frac{1}{15}$$

Hence $x = \frac{117}{15}$, $y = \frac{39}{15}$, $z = -2$.

So, the given system of linear equations are consistent as they have unique non-trivial solution.

IV. Homogeneous System of Linear Equations

Let
$$a_1x + b_1y + c_1z = 0 \quad \ldots(i)$$
$$a_2x + b_2y + c_2z = 0 \quad \ldots(ii)$$
$$a_3x + b_3y + c_3z = 0 \quad \ldots(iii)$$

be a system of Homogeneous linear equations in three variables.

$$\Rightarrow \quad \Delta_1 = \Delta_2 = \Delta_3 = 0$$

Therefore ,the system always possesses atleast one solution $x = 0$, $y = 0$, $z = 0$,which is called trivial solution, *i.e.*, this system is always consistent.

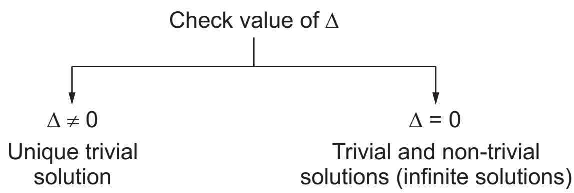


Fig. 1.1: Flowchart for homogeneous equations

> Note that if a given system of linear equations has *only zero* solutions for all its variables then the given equations are said to have *trivial solution.*

Also, note that if the system of equations $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$; $a_3x + b_3y + c_3 = 0$ is always consistent then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ but converse is NOT true.

Example 8. Find the nature of solution for the given system of equations -

$$x + y + 3z = 3, \quad 2x + 2y + 4z = 4, \quad 3x + 3y + 5z = 0$$

Solution: Here $$D = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 3 & 5 \end{vmatrix} = 0 \quad \text{But } D_1 = \begin{vmatrix} 3 & 1 & 3 \\ 4 & 2 & 4 \\ 0 & 3 & 5 \end{vmatrix} = 10$$

$\therefore$ $$D = 0 \text{ but } D_1 \neq 0, \text{ hence no solution.}$$

Video Resource References [Source: NCERT]

Determinants

Properties of determinants

Minors and co-factors

Singular and non-singular matrices

Matrix method a system of linear equations

Application of Determinants

1. Determinants apart from being used with matrices , are used in many topics like in polynomial interpolation; finding area of a parallelogram, volume of a parallelepiped; in determining number of nonidentical spanning trees in a graph etc.
2. Equation of a straight line passing through points (x_1, y_1) and (x_2, y_2) is
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$
3. Equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:
$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$
4. Area of a triangle whose vertices are (x_r, y_r); $r = 1, 2, 3$ is
$$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$. If $D = 0$ then the three points are collinear.
5. The lines
$$a_1x + b_1y + c_1 = 0 \quad ...(i)$$
$$a_2x + b_2y + c_2 = 0 \quad ...(ii)$$
$$a_3x + b_3y + c_3 = 0 \quad ...(iii)$$

are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

This is the condition for consistency of three simultaneous linear equations in 2 variables.

CASE STUDY

A farmer buys a triangular piece of land at the rate of Rs. 100 per square unit. The coordinates of the corners of the land are (0, 0), (2, 6) and (9, 10).

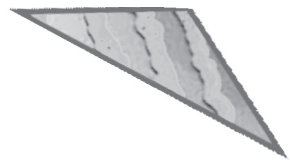


Fig. 1.2

Answer the following questions based on above information.

1. The area of the land evaluated using the determinant formula $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ is ________ sq. units.
2. If instead of taking one corner at (0, 0) the farmer would have taken it at (2, –3), what would be the area then, of the triangular plot?
3. What is the total cost paid by the farmer in lieu of buying the triangular plot?
4. What cost would farmer pay if he buys 5 pieces of land of same area as that of the triangular plot bought by him?

Check Out!!!!

After you have learnt the basics of MATLAB, download free trial version of MATLAB. (Source: Math works) Can you find out the value of the following determinants? Verify it manually!

Free trial version of MATLAB

$$A = \begin{vmatrix} 1 & 4 & 9 \\ 3 & 8 & 6 \\ 9 & 0 & 1 \end{vmatrix}$$

$$B = \begin{vmatrix} 3 & 9 & 11 \\ 20 & 1 & 0 \\ 3 & 4 & 7 \end{vmatrix}$$

Activity

- When does the Cramer's rule fail? Discuss with you teacher.
- Using any open source software, find the value of determinant.

$$\begin{vmatrix} 330 & 190 & 2947 \\ 347 & 509 & 3033 \\ 7777 & 8888 & 9999 \end{vmatrix}$$

Take print-out/screenshot and show it to your teacher. Tally your answer with your fellow students. Do you all get the same value of the determinant?

TOPIC 2 MATRICES

1.2 INTRODUCTION

A rectangular array of mn numbers (which may be real or complex) in the form of 'm' horizontal lines (called rows) and 'n' vertical lines (called columns), is called a matrix of order m by n, written as $m \times n$ matrix. Matrices can also be visualized as a set of vectors (you will study in unit 4), each row/column representing a row/column vector. Can you explain the same with the help of an example?...Go ahead and present it before your teacher!

Such an array is enclosed by [ ] or ( ) or || ||. An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

In compact form, the above matrix is represented by $A = [a_{ij}]_{m \times n}$. The numbers a_{11}, a_{12} ... etc. are known as the elements of the matrix A, a_{ij} belongs to the i^{th} row and j^{th} column and is called the $(i, j)^{th}$ element of the matrix $A = [a_{ij}]$.

Example: $A = \begin{bmatrix} 0 & -1 \\ 2 & 7 \\ 3 & 1 \end{bmatrix}$ is a matrix having 3 rows and 2 columns. Its order is 3×2 and it has 6 elements:

$$a_{11} = 0, a_{12} = -1, a_{21} = 2, a_{22} = 7, a_{31} = 3, a_{32} = 1$$

1.2.1 Types of Matrices

(1) **Row Matrix (Row Vector):** Let $A = [a_{11}\ a_{12}\ \ldots\ a_{1n}]$ *i.e.*, row matrix has exactly one row.

FUN FACT
The elements of a matrix can be anything, ranging from numbers to equations to pictures!

(2) **Column Matrix (Column Vector):** Let $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ *i.e.*, column matrix has exactly one column.

(3) **Zero or Null Matrix:** $(A = O_{m \times n})$ An $m \times n$ matrix whose all entries are zero is called a null matrix.

(4) **Horizontal Matrix:** A matrix of order $m \times n$ is a horizontal matrix if $n > m$ e.g., $\begin{bmatrix} 3 & 1 & 5 & 2 \\ 2 & 4 & 6 & 5 \end{bmatrix}$

(5) **Vertical Matrix:** A matrix of order $m \times n$ is a vertical matrix if $m > n$ e.g., $\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 9 \\ 3 & 1 \end{bmatrix}$

(6) **Rectangular Matrix:** A matrix is said to be a rectangular matrix, if the number of rows and the number of columns are not equal *i.e.*, a matrix $A = [a_{ij}]_{m \times n}$ is called a rectangular matrix, iff $m \neq n$.

(7) **Square Matrix:** A matrix is said to be a square matrix, if the number of rows and the number of columns are equal *i.e.*, a matrix $A = [a_{ij}]_{m \times n}$ is called a square matrix, iff $m = n$.

For example,

(*i*) $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 0 & 7 \\ 4 & 8 & 4 \end{bmatrix}_{3\times 3}$ (*ii*) $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

are called square matrices.

Note:

(*a*) If $A = [a_{ij}]$ is a square matrix of order n, then elements (entries) $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are said to constitute the diagonal of the matrix A. The line along which the diagonal elements lie is called principal or leading diagonal. Thus, if $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, then the elements of the diagonal of A are 2, 5.

(*b*)

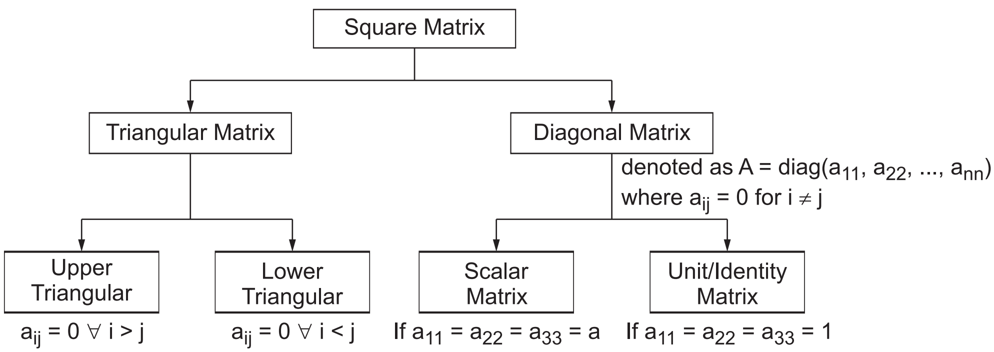


Fig. 1.3: Types of square matrices

8. **Scalar Matrix:** A diagonal matrix is said to be a scalar matrix, if its diagonal elements are equal, thus $A = [a_{ij}]_{n \times n}$ is called scalar matrix, if

$$a_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ k, & \text{if } i = j \end{cases}, \text{ where } k \text{ is scalar.}$$

9. **Sub Matrix:** A Matrix which is obtained from a given matrix by deleting any number of rows and number of columns is called a sub-matrix of the given matrix.

10. **Trace of a Matrix:** The sum of all diagonal elements of a square matrix $A = [a_{ij}]_{n \times n}$ (say) is called the trace of a matrix A and is denoted by Tr(A).

Thus $$\text{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

For example, If $A = \begin{bmatrix} 2 & -4 & 8 \\ 0 & 3 & 5 \\ 9 & 5 & 4 \end{bmatrix}$, then

$$\text{Tr}(A) = 2 + 3 + 4 = 9$$

11. **Singleton Matrix:** A matrix is said to be singleton matrix, if it has only one element *i.e.*, a matrix $A = [a_{ij}]_{m \times n}$ is said to be singleton matrix, if $m = n = 1$.
12. **Comparable Matrices:** Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are said to be comparable, if $m = p$ and $n = q$.

1.2.2 ALGEBRA OF MATRICES

FUN FACT
The term '**matrix**' was introduced by **James Sylvester**, but it was his friend & mathematician Arthur Cayley who developed the algebraic aspect of matrices.

I. Addition of Matrices

Let A, B be two matrices, each of same order $m \times n$. Then their sum $A + B$ is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A + B = [a_{ij} + b_{ij}]_{m \times n}$, $\forall\ i, j$

Example 9. Given $A = \begin{bmatrix} 1 & 7 & 3 \\ -1 & 0 & 1 \\ 0 & 5 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 7 & 1 \\ 1 & -1 & 5 \end{bmatrix}$.

Evaluate the following (whichever is defined):

(1) $A + B$ (2) $A + C$

Solution:

(1) Given A is a matrix of order 3×3 whereas B is a matrix of order 3×2 since A and B are not of the same order, sum $A + B$ is not defined.

(2) A and C, both the matrices are of the same order 3×3 therefore the sum $A + C$ is defined.

$$\therefore \quad \text{sum } A + C = \begin{bmatrix} 1 & 7 & 3 \\ -1 & 0 & 1 \\ 0 & 5 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -2 \\ 2 & 7 & 1 \\ 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1+3 & 7+4 & 3-2 \\ -1+2 & 0+7 & 1+1 \\ 0+1 & 5-1 & -3+5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 11 & 1 \\ 1 & 7 & 2 \\ 1 & 4 & 2 \end{bmatrix} \quad \textbf{Ans.}$$

Properties of Matrix Addition

Property 1: Addition of matrices is commutative,

i.e., $A + B = B + A$

Property 2: Addition of matrices is associative

i.e., $A + (B + C) = (A + B) + C$

Property 3: Existence of additive identity

i.e., $A + O = O + A$

The null matrix O is the identity element for matrix addition.

Property 4: Existence of additive inverse.

If $A + B = O = B + A$

where O is the $m \times n$ null matrix,then, the matrix B is called the additive inverse of the matrix A or the negative of A.

II. Scalar Multiplication

Let $A = [a_{ij}]_{m \times n}$ be a matrix and k be any number called a scalar. Then, the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA.

Thus, $$kA = [ka_{ij}]_{m \times n}$$

Example 10. If $A = \begin{bmatrix} 1 & 2 \\ 9 & 5 \end{bmatrix}$ and $k = 2$

Find $kA = ?$

Solution:
$$kA = 2 \times \begin{bmatrix} 1 & 2 \\ 9 & 5 \end{bmatrix}$$
$$2A = \begin{bmatrix} 2\times1 & 2\times2 \\ 2\times9 & 2\times5 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2 & 4 \\ 18 & 10 \end{bmatrix} \quad \textbf{Ans.}$$

Question: If $A = \begin{bmatrix} 0 & 1 \\ 7 & -3 \end{bmatrix}$ and $kA = \begin{bmatrix} 0 & a \\ 4b & 12 \end{bmatrix}$ then evaluate $b - a - k$. [**Ans.** 1]

III. Subtraction of Matrices

Let A, B be two matrices, each of same order $m \times n$. Then their subtraction $A - B$ is a matrix of order $m \times n$ and is obtained by subtracting the corresponding elements of A and B.

Thus, if $$A = [a_{ij}]_{m \times n} \quad \text{and} \quad B = [b_{ij}]_{m \times n}$$

Then, $$A - B = [a_{ij} - b_{ij}]_{m \times n} \ \forall \ , i, j$$

Example 11. If $A = \begin{bmatrix} 9 & 10 \\ 13 & 20 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 8 & 9 \end{bmatrix}$. Find $A - B$.

Solution: As A and B both are matrices of same order 2×2, therefore $A - B$ is well defined.

$$\therefore \quad A - B = \begin{bmatrix} 9 & 10 \\ 13 & 20 \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 9-3 & 10-5 \\ 13-8 & 20-9 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 11 \end{bmatrix} \quad \textbf{Ans.}$$

Question: Given $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 9 & 3 \\ 7 & 7 & 7 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 3 & 0 \\ 5 & 8 & 3 \\ 1 & -1 & -2 \end{bmatrix}$. Can you find out the matrix C such that

$A + 2C = B$? Evaluate! $\left[\textbf{Hint.}\ C = \frac{1}{2}[B - A]\right]$

> **Remember:** If two matrices A and B are of the same order, then only their addition and subtraction is possible and these matrices are said to be *conformable* for addition or subtraction. On the other hand, if the matrices A and B are different orders, then their addition and subtraction is not possible and these matrices are called *non-conformable* for addition and subtraction.

IV. Multiplication of Matrices (Row by Column)

Let A be a matrix of order $m \times n$ and B be a matrix or order $p \times q$, then the matrix multiplication AB is possible if and only if $n = p$ and matrices are said to be **conformable** for multiplication. In the product AB, A is called *pre-factor* and B is called *post factor*.

$\Rightarrow$ AB is possible if and only if number of columns in pre-factor = number of rows in post-factor.

Let $A_{m\times n} = [a_{ij}]$ and $B_{n\times p} = [b_{ij}]$, then order of AB is $m \times p$ and $(AB)_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj}$

For example,

Let $$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 9 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 4 \\ 8 & 1 \\ 0 & 1 \end{bmatrix}$$

Then as A is of order 2×3 and B is of order 3×2, the product is defined.

i.e., $$AB = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 9 & 2 \end{bmatrix}_{2\times 3} \times \begin{bmatrix} 3 & 4 \\ 8 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2}$$

AB is a 2×2 matrix in which each element is the sum of the products across some row of A with the corresponding entries down some column of B. These calculations are as below,

$$AB = \begin{bmatrix} \boxed{1 \;\; 3 \;\; -1} \; \boxed{\begin{matrix} 3 \\ 8 \\ 0 \end{matrix}} & \boxed{1 \;\; 3 \;\; -1} \; \boxed{\begin{matrix} 4 \\ 1 \\ 1 \end{matrix}} \\ \boxed{0 \;\; 9 \;\; 2} \; \boxed{\begin{matrix} 3 \\ 8 \\ 0 \end{matrix}} & \boxed{0 \;\; 9 \;\; 2} \; \boxed{\begin{matrix} 4 \\ 1 \\ 1 \end{matrix}} \end{bmatrix}$$

$$\Rightarrow \quad AB = \begin{bmatrix} 1\times3+3\times8+(-1)\times0 & 1\times4+3\times1+(-1)\times1 \\ 0\times3+9\times8+2\times0 & 0\times4+9\times1+2\times1 \end{bmatrix}$$

$$\Rightarrow \quad AB = \begin{bmatrix} 3+24+0 & 4+3-1 \\ 0+72+0 & 0+9+2 \end{bmatrix}$$

Thus, $$AB = \begin{bmatrix} 27 & 6 \\ 72 & 11 \end{bmatrix}_{2\times 2}$$

Question: If $A = \begin{bmatrix} 0 & 1 & 3 \\ 4 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}_{3\times 3}$, $B = \begin{bmatrix} 1 & -3 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}_{3\times 2}$. Calculate the product AB. Is BA defined? Analyze.

$$\left[\textbf{Ans. } \begin{bmatrix} 4 & 3 \\ 12 & -12 \\ 7 & -5 \end{bmatrix}_{3\times 2}\right]$$

Question: If A, B are two matrices such that $A + B = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$, $A - B = \begin{bmatrix} 7 & 2 \\ -1 & 0 \end{bmatrix}$ then find AB.

$$\left[\textbf{Ans. } \begin{bmatrix} -10 & -2 \\ 1 & 3 \end{bmatrix}\right]$$

Properties of Matrix Multiplication

1. *Matrix multiplication is not commutative, i.e., $AB \neq BA$ (in general)*

   Example: *Here both AB and BA exist and also they are of the same type but $AB \neq BA$*

   Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

   $\Rightarrow \quad AB \neq BA$

2. *$AB = 0$ does not imply that either $A = 0$ or $B = 0$*

   For ex: Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ *(non-zero) and* $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ *(non-zero)*

   then $AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

   Note: If A and B are two non-zero matrices such that $AB = 0$ then A and B are called the divisors of zero.

3. *Matrix multiplication is associative*

   If A, B and C are conformable for the product AB and BC, then $(AB)C = A(BC)$

   Distributivity:

   $$\left.\begin{array}{l} A(B+C) = AB + AC \\ (A+B)C = AC + BC \end{array}\right]$$ *provided A, B and C are conformable for respective products.*

4. *For a square matrix A,* $A^n = \underbrace{A \cdot A \cdot A \ldots A}_{\text{upto } n \text{ times}}$ *where* $n \in N$

1.2.3 The Transpose of a Matrix (Changing Rows and Columns)

Let $A = [a_{ij}]$ be any matrix of order $m \times n$. Then A^T or $A' = [a_{ji}]$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ of order $n \times m$.

Example 12. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Find the values of θ satisfying the equation $A^T + A = I_2$.

Solution: We have, $A^T + A = I_2$

$$\Rightarrow \quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad \theta = 2n\pi + \frac{\pi}{3}, \; n \in N$$

1.2.4 Orthogonal Matrix

A square matrix A is said to be orthogonal matrix, iff $AA' = I$, where I is an identity matrix

Note:

1. If $AA' = I$, then $A^{-1} = A^T$
2. If A and B are orthogonal, then AB is also orthogonal.
3. If A is orthogonal, then A^{-1} and A' are also orthogonal
4. The determinant value of orthogonal matrix is either 1 or –1.

Example 13. Test the matrix $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ for orthogonality.

Solution: A matrix A is orthogonal iff $AA' = I$.

Here, $$A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

Now, $$AA' = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \quad AA' = \begin{bmatrix} 4+0 & 0 \\ 0 & 0+4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\Rightarrow \quad AA' = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow AA' = 4I$$

Hence, A is not orthogonal.

Remark: If $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ then it is orthogonal. Prove and check it yourself!

1.2.5 Symmetric and Skew Symmetric Matrix

Definition: Symmetric matrix - A square matrix $A = [a_{ij}]$ is said to be symmetric if, $a_{ij} = a_{ji}\ \forall\ i$ and j. So, for a symmetric matrix $A = A^T$.

Definition: Skew symmetric matrix - A Square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $a_{ij} = -\, a_{ji}\ \forall\ i$ and j. For a skew symmetric matrix $A = -\, A^T$.

For example, If $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, then $A^T = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$. Here, A is symmetric matrix as $A^T = A$

Now consider the following example:

If $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix} = -\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix} = -\, A$

Here, A is skew symmetric matrix as $A^T = -\, A$.

Remember:

(*i*) If A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \; \forall \; i$. Thus the diagonal elements of a skew-symmetric square matrix are all zero, but not the converse.

(*ii*) Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix..

$$A = \underbrace{\frac{1}{2}(A+A^T)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A-A^T)}_{\text{Skew symmetric}} \quad \text{and} \quad A = \frac{1}{2}(A^T+A) - \frac{1}{2}(A^T - A)$$

Example 14. If the square matrix $B = [b_{ij}]_{m \times n}$ is given by $b_{ij} = (i-j)^n$, show that B is symmetric or skew-symmetric matrix according to as n is even or odd respectively.

Solution:

$$b_{ij} = (i-j)^n = (-1)^n (j-i)^n$$

$$= (-1)^n b_{ji} = \begin{cases} b_{ji}, & n \text{ is even integer} \\ -b_{ji}, & n \text{ is odd integer} \end{cases}$$

Hence, A symmetric if n is even and skew symmetric if n is odd integer.

Example 15. Can you express A as the sum of a symmetric and a skew symmetric matrix, where $A = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$. Explain with proof.

Solution: Yes we can express A as the sum of a symmetric and a skew symmetric matrix.

We have, $$A = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$$

$$P = \frac{1}{2}(A+A^T) = \frac{1}{2}\begin{bmatrix} 2 & 7 \\ 7 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 7/2 \\ 7/2 & -2 \end{bmatrix} = P^T$$

Thus, $$P = \frac{1}{2}(A+A') \text{ is a symmetric matrix.}$$

Also let $$Q = \frac{1}{2}(A-A^T) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow Q^T = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} = -Q$$

Thus $$Q = \frac{1}{2}(A-A^T) \text{ is a skew- symmetric matrix.}$$

Now, $$P+Q = \begin{bmatrix} 1 & 7/2 \\ 7/2 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = A$$

Hence, A is represented as the sum of a symmetric and a skew symmetric matrices.

Question. If A is symmetric as well as skew symmetric matrix, then A is:

(A) diagonal matrix (B) null matrix (C) triangular matrix (D) identity matrix.

[**Ans.** (B)]

1.2.6 Singular and Non-Singular Matrices

A square matrix A is said to be a *singular*, if $|A| = 0$ and a square matrix A is said to be *non-singular*, if $|A| \neq 0$.

Example 16:

(*i*) $A = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ is a singular matrix,since $|A| = 0$.

(*ii*) $A = \begin{bmatrix} 5 & 3 \\ 4 & 5 \end{bmatrix}$ is a non-singular matrix, since $|A| = 25 - 12 = 13 \neq 0$

1.2.7 Inverse of a Matrix

I. Adjoint of a Square Matrix

Definition:

Let $A = [a_{ij}]$ be some $n \times n$ matrix. The transpose B' *i.e.*, $B = [A_{ij}]_{n \times n}$
where, A_{ij} denotes the cofactor of the element a_{ij} in the determinant $|A|$ is called the adjoint of the matrix A and is denoted by the symbol Adj. A.

Thus the adjoint of a matrix A is the transpose of the matrix formed by the cofactors of A *i.e.*, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then $$\text{adj } A = \text{the transpose of the matrix } \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

$$= \text{the matrix } \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Note: (*i*) Sometimes the adjoint of a matrix is also called the 'adjugate' of the matrix and $|\text{adj } A| = |A|^{n-1}$.

(*ii*) $A(\text{adj } A) = (\text{adj } A)A = |A|I$.

Rule to Write Cofactors of an Element

Cross the row and column intersection at the element a_{ij} and the determinant which is left to be denoted by D.

Then, $$\text{cofactors of } a_{ij} = \begin{cases} D, & \text{if } i + j = \text{even integer} \\ -D, & \text{if } i + j = \text{odd integer} \end{cases}$$

Example 17. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, then find Adj. A.

Solution: In $|A|$, the cofactor of α is δ and the cofactor of β is $-\gamma$. Also the cofactor of γ is $-\beta$ and the cofactor of δ is α. Therefore the matrix B formed of the cofactors of the elements of $|A|$ is

$$B = \begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix}$$

Now $$\text{Adj } A = \text{the transpose of the matrix } B$$

$$= \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$$

Example 18. Find the adjoint of the matrix.

$$A = \begin{bmatrix} -2 & 2 & 3 \\ -1 & 5 & 10 \\ 4 & 4 & 2 \end{bmatrix}$$

Solution: We have $$|A| = \begin{bmatrix} -2 & 2 & 3 \\ -1 & 5 & 10 \\ 4 & 4 & 2 \end{bmatrix}$$

The cofactors of the elements of the first row of the determinants

$|A|$ are $\begin{vmatrix} 5 & 10 \\ 4 & 2 \end{vmatrix}, -\begin{vmatrix} -1 & 10 \\ 4 & 2 \end{vmatrix}, \begin{vmatrix} -1 & 5 \\ 4 & 4 \end{vmatrix}$, *i.e.*, are – 30, 42, – 24 respectively.

The cofactors of the elements of the second row of the determinants

$|A|$ are $-\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}, \begin{vmatrix} -2 & 3 \\ 4 & 2 \end{vmatrix}, -\begin{vmatrix} -2 & 2 \\ 4 & 4 \end{vmatrix}$, *i.e.*, are 8, – 16, 16 respectively.

The cofactors of the elements of the third row of the determinants

$\begin{vmatrix} 2 & 3 \\ 5 & 10 \end{vmatrix}, -\begin{vmatrix} -2 & 3 \\ -1 & 10 \end{vmatrix}, \begin{vmatrix} -2 & 2 \\ -1 & 5 \end{vmatrix}$, *i.e.*, 5, 17, – 8

Therefore the Adj. A= the transpose of the matrix B where

$$B = \begin{bmatrix} -30 & 42 & -24 \\ 8 & -16 & 16 \\ 5 & 17 & -8 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} -30 & 8 & 5 \\ 42 & -16 & 17 \\ -24 & 16 & -8 \end{bmatrix}$$

Question: Find the adjoint of the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix} \text{ and verify the theorem.}$$

$$A(\text{adj } A) = (\text{adj } A)A = |A|I$$

$\left[\textbf{Ans. } \begin{bmatrix} 6 & -4 \\ -1 & 3 \end{bmatrix} \right]$

Question: Verify that the adjoint of a diagonal matrix of orders 3 is a diagonal matrix.

Question: If O be a zero matrix of order n, then analyse whether adj. $O = O$ is true or false.

Question: If I_n be a matrix of order n, then can you show that adj $I_n = I_n$?

II. Inverse of a Matrix (Reciprocal Matrix)

A square matrix A (non-singular) of order n is said to be invertible, if there exists a square matrix B of the same order such that $AB = I_n = BA$.

Then B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus, $A^{-1} = B \Leftrightarrow I_n = BA$

> **FUN FACT**
> English mathematician James Sylvester met fellow mathematician Arthur Cayley while studying law!

We have, $\quad A(\text{adj } A) = |A|I_n$

$\Rightarrow \quad A^{-1} A(\text{adj A}) = A^{-1} I_n|A| \quad \Rightarrow \quad I_n(\text{adj } A) = A^{-1}|A|I_n$

$\therefore \quad A^{-1} = \dfrac{\text{adj } A}{|A|}$; provided $|A| \neq 0$

Note: The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Properties of Inverse

(*i*) *If A and B are invertible matrices of the same order, then* $(AB)^{-1} = B^{-1} A^{-1}$

(*ii*) *If A be an invertible matrix, then* A^T *is also invertible and* $(A^T)^{-1} = (A^{-1})^T$

(*iii*) *If A is invertible, then (a)* $(A^{-1})^{-1} = A$ *and (b)* $(A^k)^{-1} = (A^{-1})^k$; $k \in N$

(*iv*) *If A is non-singular matrix, then* $|A^{-1}| = |A|^{-1}$

(*v*) *Orthogonal matrix A is always invertible and* $A^{-1} = A^T$

> **Important Note:** For the products AB, BA to be both defined and be equal, it is necessary that A and B are both square matrices of the same order. Thus non-square matrices cannot possess inverse.

Example 19. Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Given that $\quad |A| \neq 0 \quad$ and $\quad |A| = 2$

Solution: Since $|A| \neq 0$, therefore the matrix A is non-singular and possesses inverse.

Now the cofactors of the elements of the first row of the determinant $|A|$ are

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}, \textit{ i.e.}, -1, 1, 1$$

Second row $-\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$, *i.e.*, $1, -1, 1$

Third row $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$, *i.e.*, $1, 1, -1$

Then,
$$B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \Rightarrow B' = (\text{Adj } A) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \text{Adj}(A) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Note: After finding the inverse of a matrix A, we must check our answer by verifying the relation $A \cdot A^{-1} = I$

Example 20. If the matrices A and B commute, then prove that A^{-1} and B^{-1} also commute.

Solution: Since A and B commute, therefore $AB = BA$

Now $(AB)^{-1} = B^{-1} A^{-1}$

Also $(AB)^{-1} = (BA)^{-1} = A^{-1} B^{-1}$

$\therefore$ $B^{-1} A^{-1} = A^{-1} B^{-1}$

Thus A^{-1} and B^{-1} also commute

Example 21. If A, B, C be three matrices conformable for multiplication , then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Solution: We have $(ABC)^{-1} = \{A(BC)\}^{-1} = (BC)^{-1}A^{-1} = (C^{-1}B^{-1})A^{-1} = C^{-1}B^{-1}A^{-1}$

Example 22. Prove that if a matrix A is non-singular, then $AB = AC$ implies $B = C$, where B and C are square matrices of the same order as A.

Solution: Since the matrix A is non-singular, therefore A^{-1} exists

Hence $AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow$ $(A^{-1}A)B = (A^{-1}A)C \Rightarrow I_n B = I_n C$

$\Rightarrow$ $B = C$

Example 23. Given $AB = AC$. Does it follow that $B = C$? Can you provide a counter example?

Solution: If the matrix A is non-singular then $AB = AC$ implies $B = C$. But if the matrix A is singular, then $AB = AC$ does not necessarily imply $B = C$ the following example will make it clear.

Let
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

We have $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = AC$, though $B \neq C$

Question: Find the inverse of the matrix

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$ Also verify your result.

$$\left[\textbf{Ans. } A^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\right]$$

Question: Given that $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ compute.

(*i*) det A (*ii*) Adj A (*iii*) A^{-1}

$$\left[\textbf{Ans. } \det A = 2;\ A^{-1} = \frac{1}{2}\begin{vmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{vmatrix}\right]$$

1.2.8 Matrix Method

(Use of the inverse of a matrix to find the solution of a system of linear equations)

Consider a system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$.

i.e., $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i,\ i = 1, 2, \ldots, n.$

These equations can be written in the form of a single matrix equation $AX = B$.

Where,
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n},\quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}_{n\times 1},\quad B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_n \end{bmatrix}$$

Suppose A is a non-singular matrix

i.e., $|A| \neq 0$

Then A^{-1} exists and $X = A^{-1}B$.

Types of Equations and their Consistency

Type 1: For non-homogeneous system of equations if

(*a*) $|A| \neq 0$, then the system of equation is consistent and has a unique solution given by $X = A^{-1}B$.

(*b*) $|A| = 0$ and (adj A). $B \neq O$ then the system of equations is inconsistent and has no solution.

(*c*) $|A| = 0$ and (adj A) $\cdot$ $B = O$, then the system of equations is consistent and has an infinite number of solutions.

Type 2: For homogeneous system of equations if

(*a*) $|A| \neq 0$, then the system of equations has only trivial solution and it has one solution.

(*b*) $|A| = 0$, then the system of equations has non-trivial solution and it has infinite solutions.

(*c*) Number of equations < number of unknowns, then it has non-trivial solution.

Note: Non-homogeneous linear equations can also be solved by Cramer's rule, this method has been discussed in the topic on determinants.

Example 24. Write down in matrix form the system of equation

$$\begin{aligned} x + y + z &= 92 \\ 2x + 5y + 7z &= 52 \\ 2x + y - z &= 10 \text{ and find } A^{-1}, \text{ if} \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \text{ and hence solve the given equation}$$

Solution: The given system of equation can be written in matrix form as

$$AX = B \qquad ...(1)$$

where,
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 92 \\ 52 \\ 10 \end{bmatrix}$$

We have $|A| = 1(-5-7) - 1(-2-14) + 1(2-10) = -12 + 16 - 8 = -4$

Therefore A is non-singular and thus A^{-1} exists. Let us now find A^{-1}

The cofactors of the first row of $|A|$ are

$$A_{11} = \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} = -5 - 7 = -12 \qquad A_{12} = -\begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} = -(-2-14) = 16$$

$$A_{13} = \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} = 2 - 10 = -8$$

The cofactors of the second row of $|A|$ are

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -(-1-1) = 2 \qquad A_{22} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(1-2) = 1$$

The cofactors of the third row of $|A|$ are

$$A_{31} = \begin{vmatrix} 1 & 1 \\ 5 & 7 \end{vmatrix} = 7 - 5 = 2 \qquad A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} = -(7 - 2) = -5$$

$$A_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} = 5 - 2 = 3$$

$\therefore$ Adj A = the transpose of the matrix B where

$$B = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

$$\Rightarrow \quad \text{Adj } A = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{Adj } A = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$\therefore$ Now pre-multiplying (1) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$
$$I_3X = A^{-1}B$$
$$X = A^{-1}B$$

We have $$A^{-1}B = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 92 \\ 52 \\ 10 \end{bmatrix}$$

$\Rightarrow$ $$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -12\times 92 + 2\times 52 + 2\times 10 \\ 16\times 92 + (-3)\times 52 + (-5)\times 10 \\ -8\times 92 + 1\times 52 + 3\times 10 \end{bmatrix}$$

$$= \frac{-1}{4}\begin{bmatrix} -1104+104+20 \\ 1472-156-50 \\ -736+52+30 \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -980 \\ 1266 \\ -654 \end{bmatrix} = \begin{bmatrix} 245 \\ -\frac{633}{2} \\ \frac{327}{2} \end{bmatrix}$$

$$x = 245, \quad y = \frac{-633}{2}, \quad z = \frac{327}{2}$$ **Ans.**

Example 25. Prove that if A is non-singular matrix such that A is symmetric then A^{-1} is also symmetric.

Solution: $A^T = A$ [$\because$ A is a symmetric matrix]

$(A^T)^{-1} = A^{-1}$ [Since A is a non-singular matrix]

$\Rightarrow$ $(A^{-1})^T = A^{-1}$ Hence proved.

Question: Solve the system using matrix method

$$x + y + z = 6$$
$$x - y + 2z = 7$$
$$2x - y + 3z = 12$$

[**Ans.** $x = 2, y = 1, z = 3$]

Question: Solve the system $x + y + z = 6$, $x - y + z = 2$, $2x + y - z = 1$ using matrix method.

[**Ans.** $x = 1, y = 2, z = 3$]

Example 26. Solve the system of equations

$$x + 4y + 7z = 0$$
$$2x + 5y + 8z = 0$$
$$3x + 6y + 9z = 0$$

Solution: We have $$x + 4y + 7z = 0$$
$$2x + 5y + 8z = 0$$
$$3x + 6y + 9z = 0$$

The given system of equation in the matrix form are written as below

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = O$$

where, $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore \quad |A| = 1(45 - 48) - 4(18 - 24) + 7(12 - 15)$$
$$= -3 + 24 - 21 = 0$$
$$|A| = 0$$

and therefore the system has a non-trivial solution. Now we may write first two of the given equations

$$x + 4y + 7z = 0 \quad \text{and} \quad 2x + 5y + 8z = 0.$$

Solving these equation in terms of x ,we get,

$$x = -(4y + 7z), \quad 2x = -(5y + 8z)$$

i.e., $$x = -\frac{5y+8z}{2} = -(4y + 7z), \text{ which gives } y = -2x.$$

Now putting $z = x$ and $y = -2x$ in third Eq. of the given system $(3x + 6y + 9z = 0)$

We get $$\text{LHS} = 3x + 6(-2x) + 9(x) = 3x - 12x + 9x = 0 \text{ RHS}.$$

Hence, third equation is satisfied by $z = x$ and $y = -2x$

Now, let $$\frac{z}{1} = \frac{y}{-2} = \frac{x}{1} = k$$

Hence $x = k$, $y = -2k$, $z = k$ (where k is arbitrary constant)

Hence the equation has infinite number of solutions.

Video Resource References [Source: NCERT]

Introduction to matrices

Types is matrices

Operation on matrices part-1

Matrices and determinants-an introduction

Properties of matrix

Transpose of a matrix

Application of Matrices

Matrix instead of being a quantity can be considered as an operator to be used for transformation of vector or variables. The addition and multiplication of matrices are compositions of operators instead of scalar quantities.

Geometrical Applications

(1) Reflection: If the point (x, y) is reflected by the y-axis then new coordinates are $X' = -x$, $Y' = y$ ... (1) can be written in form

$$X' = -1x + 0.y \quad \text{and} \quad Y' = 0.x + 1y$$

In terms of matrix they can be written as

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Hence the reflection about y-axis is obtained by pre multiplying the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

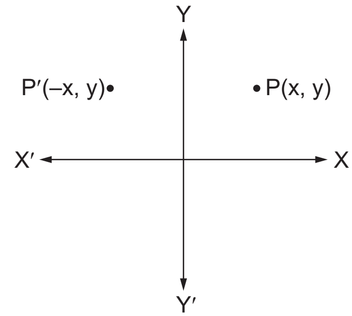


Fig. 1.4: Reflection of a point about y-axis

Similarly the reflection in x-axis is obtained by pre-multiplying with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(2) **Reflection about the line $y = x$**

Since $\quad y' = x, \quad x' = y$

And so $\quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Similarly the reflection of (x, y) about the line $y = -x$ is obtained by $x' = -y$, $y' = -x$

or $\quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

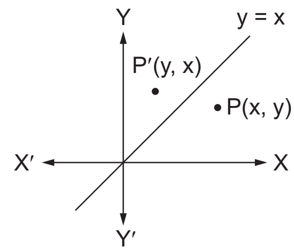


Fig. 1.5: Reflection about the line y = x

(3) Rotation about origin: If the coordinates of a point A are (x, y), O being the origin and the line OA is rotated about O as centre through an angle α in the anti-clockwise direction then the new coordinates of A are

$$X' = r\cos(\theta + \alpha) = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha = x\cos\alpha - y\sin\alpha$$
$$Y' = r\sin(\theta + \alpha) = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha = y\cos\alpha + x\sin\alpha$$

So that $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ where (r, θ) be the polar coordinates of (x, y). If rotation is in clockwise direction we put $-\alpha$ for α.

General Applications

(1) In robotics and automation matrices are the basic component for the robot movement.

(2) Matrix help in the calculation of battery power output, register, and conversion of electrical energy into other useful energy.

(3) Matrix are used to alter the object in 3D space. It can help to make animation more precise and perfect.

(4) Matrices are used in the study of electrical circuit, quantum mechanics and optics.

(5) Hessian matrices, Jacobians etc play important role in fields like operation research.

Interesting Fact: Matrices are so popular that many movies have been named after them!!!!

SUMMARY

1. **Determinants** are defined as a scalar values, denoted by det A or $|A|$ or Δ.
2. **Minor:** The minor of a given element of determinant is the determinant obtained by deleting the row and column in which the given element stands.
3. **Cofactor:** If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element is given by $C_{ij} = (-1)^{i+j} M_{ij}$
4. If $A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $B = \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}$.

   Then, $A \times B = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$
5. System of linear equations (Cramer's Rule)in two variables

$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

6. System of linear equations in three variables (Cramer's Rule)

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta}, \text{ where } \Delta \neq 0$$

7. A rectangular array of mn numbers (real or complex) in the form of 'm' horizontal lines (called rows) and 'n' vertical lines (called columns), is called a matrix of order m by n, written as $m \times n$ matrix.
8. **Row matrix (Row vector):** Row matrix has exactly one row.
9. **Column matrix (Column vector):** Column matrix has exactly one column.
10. **Zero or Null matrix:** ($A = O_{m \times n}$) An $m \times n$ matrix whose all entries are zero is called a null matrix.
11. Horizontal matrix: A matrix of order $m \times n$ is a horizontal matrix if $n > m$
12. **Vertical matrix:** A matrix of order $m \times n$ is a vertical matrix if $m > n$
13. **Rectangular matrix:** A matrix is said to be a rectangular matrix, if the number of rows and the number of columns are not equal *i.e.*, a matrix $A = [a_{ij}]_{m \times n}$ is called a rectangular matrix, iff $m \neq n$
14. **Square matrix:** A matrix is said to be a square matrix, if the number of rows and the number of columns are equal *i.e.*, a matrix $A = [a_{ij}]_{m \times n}$ is called a square matrix, iff $m = n$.

15. **The transpose of a matrix:** Let $A = [a_{ij}]$ be any matrix of order $m \times n$. Then A^T or $A' = [a_{ji}]$ for $1 \le i \le m$ and $1 \le j \le n$ of order $n \times m$.
16. A square matrix A is said to be orthogonal matrix, iff $AA' = I$, where I is an identity matrix
17. A square matrix A (non-singular) of order n is said to be invertible, if there exists a square matrix B of the same order such that $AB = I_n = BA$.
18. In symmetric and skew-symmetric matrices, $A = A^T$ and $A = -A^T$ respectively.
19. $A^{-1} = \dfrac{\text{adj } A}{|A|}$; $|A| \neq 0$
20. **Matrix Method:** $X = A^{-1}B$

Exercises

Subjective Questions

Q.1. Find the value of a, b, c and d so that the matrices A and B may be equal

where $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 3 \\ 9 & -4 \end{bmatrix}$$ [**Ans.** $a = 7$, $b = 3$, $c = 9$, $d = -4$]

Q.2. If $X = \begin{bmatrix} -3 & -4 \\ 7 & 3 \end{bmatrix}$, $Y = \begin{bmatrix} 6 & 9 \\ -12 & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 2 \\ 2 & -6 \end{bmatrix}$, verify that $X + (Y + Z) = (X + Y) + Z$

Q.3. If D is an $m \times n$ matrix then show that $D = -(-D)$

Q.4. If $X = \begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}$ and $Y = \begin{bmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{bmatrix}$

Show that $XY = YX$

Q.5. Find the adjoint of the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

$$\left[\textbf{Ans. } \text{Adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}\right]$$

Q.6. Find the inverse of the matrix

$$A = \begin{bmatrix} u+iv & w+id \\ -w+id & u-iv \end{bmatrix} \text{ If } u^2 + v^2 + w^2 + d^2 = 1$$

$$\left[\textbf{Ans. } \begin{bmatrix} u-iv & -w-id \\ w-id & u+iv \end{bmatrix}\right]$$

Q.7. (*a*) Show that if A is a non-singular matrix then $\det(A^{-1}) = (\det A)^{-1}$.

(*b*) If B is non-singular, prove that the matrices A and $B^{-1}AB$ have the same determinant, A and B being both square matrix of order n.

Q.8. If A is a square matrix, then is it possible that adj $A^T = (\text{adj } A)^T$? If yes, prove it!

Q.9. If A is a symmetric matrix, then prove that adj A is also symmetric.

Objective Questions

Q.1. If $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ then $\begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix}$ is equal to

(*a*) A (*b*) $3A$ (*c*) $9A$ (*d*) 81A [**Ans.** (*c*)]

Q.2. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix} =$

(*a*) 1 (*b*) 0 (*c*) $3adb$ (*d*) ab [**Ans.** (*d*)]

Q.3. If $[4 \quad z \quad 6]\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 3 & 4 \end{bmatrix}\begin{bmatrix} z \\ 1 \\ -4 \end{bmatrix} = 0$, then the value of z is:

(*a*) −11.8 (*b*) 0 (*c*) −13 (*d*) 2 [**Ans.** (*a*)]

Comprehension

Q.4. Consider the system of equations

$$2\alpha + u\beta + 6\gamma = 8, \quad \alpha + 2\beta + v\gamma = 5 \quad \text{and} \quad \alpha + \beta + 3\gamma = 4 \text{ then}$$

(*i*) This system of equation will have infinite solution if

(*a*) $v = 32$ (*b*) $v = 12$ (*c*) $v = 13$ (*d*) $v = 2$

(*ii*) The given system of equation will have no solution if

(*a*) $u = 3, v \neq 4$ (*b*) $u \neq 13, v = 32$ (*c*) $u \neq 2, v = 3$ (*d*) $u = 12, v = 32$

(*iii*) The given system of equations will have unique solution if

(*a*) $u \neq 2, v \neq 3$ (*b*) $u \neq 3$ (*c*) $u = 2, v = 3$ (*d*) $u = 2, v \neq 3$

[**Ans.** (*i*) (*d*), (*ii*) (*c*), (*iii*) (*a*)]

Q.5. Let $\begin{vmatrix} 1+\alpha & \alpha & \alpha^2 \\ \alpha & 1+\alpha & \alpha^2 \\ \alpha^2 & \alpha & 1+\alpha \end{vmatrix} = a\alpha^5 + b\alpha^4 + c\alpha^3 + d\alpha^2 + e\alpha + f$

Match the entries from the following columns

	Column-I	Column-II	
(*A*)	The value of "f" is equal to	(*a*)	0
(*B*)	The value of "e" is equal to	(*b*)	1
(*C*)	The value of $a + c$ is equal to	(*c*)	−1
(*D*)	The value of $b + d$ is equal to	(*d*)	3

[**Ans.** (*A*) → (*b*); (*B*) → (*d*); (*C*) → (*c*); (*D*) → (*a*)]

Q.6. If 'B' is 3×3 matrix and det $(3B) = R\{\det(B)\}$, R is equal to

(*a*) 7 (*b*) 1 (*c*) 6 (*d*) 27 [**Ans.** (*d*)]

Q.7. If X, Y are square matrix of order 3, such that $|X| = -1$, $|Y| = 3$ then $|XY|$ is equal to

(*a*) 81 (*b*) −81 (*c*) 18 (*d*) −18 [**Ans.** (*d*)]

Q.8. If $D = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ then adj D equal to

(*a*) D (*b*) D^T (*c*) $3D$ (*d*) $3D^T$ [**Ans.** (*d*)]

Q.9. If 'L' is a 3×3 matrix and "M" is its adjoint such that $|M| = 64$, then $|L|$ is equal to

(*a*) ± 16 (*b*) ± 64 (*c*) ± 8 (*d*) ± 4 [**Ans.** (*c*)]

Q.10. For any 2×2 matrix D, if the $D(\text{adj } D) = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, then $|D|$ is equal to _______. [**Ans.** 5]

Q.11. If D is a singular matrix, then adj D is _______. [**Ans.** not defined]

Q.12. If D is non singular matrix, then $D(\text{adj } D)$ = _______. [**Ans.** ($|D|I$)]

Q.13. The inverse of matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is _______. $\left[\textbf{Ans. } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right]$

Q.14. If a matrix A is both symmetric and skew-symmetric than A is a _______ matrix. [**Ans.** null]

CASE STUDY

Three polytechnic colleges A, B and C organized a fund raising activity for helping the patients affected by a pandemic in need. They sold Neem, Peepal and Money plants at a cost of ₹ 25, ₹ 25, and ₹ 20, respectively. The number of plants sold by each polytechnic college A, B, C are given below.

Plant	Poly		
	A	B	C
Neem	100	90	70
Peepal	70	120	55
Money	20	30	50

Based on above information, answer the following questions:

(1) The total fund raised for the pandemic is ______.

(2) The funds collected by the three polytechnic colleges arranged in increasing order are
₹ ______ [College ______] ≥ ₹ ______ (College ______) ≥ ₹ ______ (College ______)

(3) If F be a 1×3 matrix representing the sale price of each plant per piece, then F = ?

(4) If T be a 3×3 matrix in which the columns represent the 3 polytechnic colleges and rows represent the sale of plants, then T = ?

Check Out!!!!

After having learnt the basics of MATLAB, download free trial version of MATLAB. (Source: Math works) Checkout whether you can find the inverse of the matrix

$$A = \begin{vmatrix} 3 & 7 & 9 \\ 4 & 7 & 9 \\ 9 & 7 & 21 \end{vmatrix}$$

Verify Your Result Manually!

Activity

Adjacency Matrix: It is a square matrix which is used to represent a finite graph especially in the field of Graph theory and computer science. Find out the definition of adjacency matrix and explain it by taking an example of an undirected graph.

Mini Project

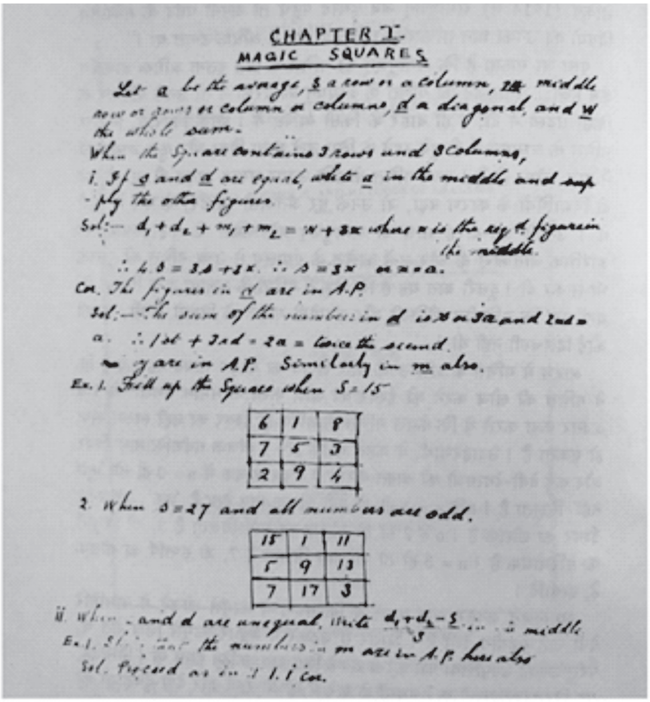
Form small groups of 4-5 students and do an online/offline survey on uses of matrices/determinants. List with details at least 10 uses. Make a verbal group presentation, present it before your subject teacher.

Know More

- **Similarity of Matrices:** Let A and B be two square matrices of order n. Then B is said to be similar to A if there exists a non-singular matrix P such that $B = P^{-1}AP$.
- Rank of a matrix is an important concept used in various applications like- with the augmented matrix for solving a system of linear equations etc.

References and Suggested Readings

- Narayan Shanti and Mittal P.K., (1953). A textbook on matrices. S. Chand and Co.
- NCERT. Mathematics Textbook Class XII (Part I)
- Lipschutz S. and Lipson M.L. Schaum's (2001) Linear Algebra. Tata Mc.Graw-Hill.
- Vivek Sahai, Vikas Bist (2001). Linear Algebra, Narosa Publishing House.



CHAPTER I.

MAGIC SQUARES

Let a be the average, s a row or a column, m middle row or column, d a diagonal, and w the whole sum.

When the Square contains 3 rows and 3 columns,

i. If s and d are equal, write a in the middle and supply the other figures.

Sol:— $d_1 + d_2 + m_1 + m_2 = w + 3x$ where x is the fig. in the middle.

$\therefore 4s = 3s + 3x \quad \therefore s = 3x$ or $x = a$.

Cor. The figures in d are in A.P.

Sol:— The sum of the numbers in d is s or $3a$ and 2nd = a. $\therefore$ 1st + 3rd = $2a$ = twice the second. $\therefore$ they are in A.P. Similarly in m also.

Ex. 1. Fill up the Square when $S = 15$

6	1	8
7	5	3
2	9	4

2. When $s = 27$ and all numbers are odd.

15	1	11
5	9	13
7	17	3

ii. When s and d are unequal, write $\frac{d_1 + d_2 - s}{3}$ in middle

Ex. 1. Show that the numbers in m are in A.P. here also

Sol. Proceed as in i. 1. i Cor.

Indian Mathematician **Srinivasa Ramanujan** (1887-1920) known as *the man who knew infinity* independently compiled nearly 3,900 results and almost all his claims have been proven correct!

An excerpt from his notebook is given here. (**Source:** Muley Gunakar (1992), Sansaar ke Mahan Ganitagya, Raajkamal Prakashan)

2 Integral Calculus

UNIT SPECIFICS

In this unit we have given a detailed self-explanatory theory on the following topics - integration as inverse operation of differentiation; simple integration by substitution, by parts and by partial fractions (for linear factors only); use of formulas $\int_0^{\pi/2} \sin^n x dx$, $\int_0^{\pi/2} \cos^n x dx$ and $\int_0^{\pi/2} \sin^m x \cos^n x dx$.

Applications of integration on area bounded by a curve and axes; and volume of a solid formed by revolution of an area about axes are discussed in a simple and lucid manner.

RATIONALE

Calculus is the mathematical language of engineers, scientists, technical professionals etc. Work of these persons ranging from your fridge, mobiles, TV and vehicles to medicines, robotics, national security leave a great impact on our lives. Integral calculus helps in determining the sum or total size or value of the object under study like length, area and volume. For instance, when we integrate a velocity function we get a distance function, which helps us to calculate the distance travelled by an object over an interval of time. It is important to mention here that Fundamental theorem of Calculus relates integration to differentiation.

Pre-requisites

- Basics of differentiation
- Knowledge of basic operations with algebraic, trigonometric and exponential functions.

Unit 2 Integral Calculus	Unit Outcomes(UO) The students will learn -
U2-O1	Use integration as the inverse operation of differentiation.
U2-O1	Apply various techniques of integration for finding anti-derivatives, analyzing them and solving problems.
U2-O3	Apply formulae $\int_0^{\pi/2} \sin^n x dx$, $\int_0^{\pi/2} \cos^n x dx$ and $\int_0^{\pi/2} \sin^m x \cos^n x dx$ in solving problems.
U2-O4	Develop a conceptual relationship between integral and area. Evaluating simple problems on area especially bounded by a curve and axes.
U2-O5	Calculate simple problems related to the volume of solids formed by revolution especially about axes with integration.

CO-UO Mapping

Unit 2 Outcomes	Expected mapping with course outcomes (1 - Weak correlation; 2 - Medium correlation; 3 - Strong correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U2-O1		3		2	1
U2-O2		3		1	1
U2-O3		1			1
U2-O4		3			1
U2-O5		3			1

2.1 INTRODUCTION

In the present unit we shall study the concept of Integral calculus, which is based on the idea of finding anti-derivatives. You have studied at length differentiation and the concept of derivative in previous semester. Now, the curiosity of finding out a function whenever its derivative is given, leads to a detailed study of integral calculus.

Integral calculus can further be studied under two parts:

1. Indefinite Integrals
2. Definite Integrals

In this unit, first we will study Integral Calculus which helps in determining functions from their derivatives. Thereafter, we will briefly understand the concept of definite integrals so as to apply them for evaluating area and volume.

2.1.1 Integration as an Inverse Operation of Differentiation

We consider the process of reversing differentiation. That is, we will take a function $f(x)$, and think of all possible functions $F(x)$ which would have $f(x)$ as their derivative. This very thought process leads to the concept of anti-derivative and integration. As by now, you already know how to find out derivatives. So let us consider the function $F(x) = 4x^3 + 9x + 11$, differentiating $F(x)$, we get

$$f(x) = F'(x) = 12x^2 + 9, \textit{ i.e.}, \text{ figuratively (Fig. 2.1) it is}$$

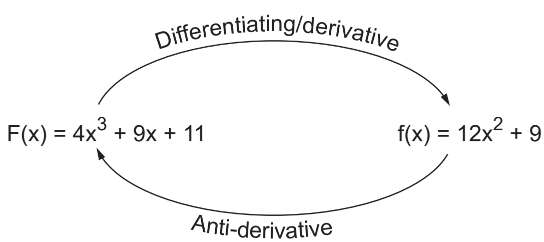


Fig. 2.1

But now the question arises whether the anti-derivative of $f(x)$ as shown above is unique or not? Let us see! These are other functions like - $4x^3 + 9x + 20$, $4x^3 + 9x + 200$, $4x^3 + 9x$ etc. which have $f(x) = 12x^2 + 9$ as their derivative. This is because the constant term in each of these functions disappears the moment we do differentiation. So, all of these are anti-derivatives of $12x^2 + 9$. In other words, if $F(x)$ is an antiderivative of $f(x)$, then $F(x) + c$ (for any constant c) is also an antiderivative for $f(x)$. Here now comes the definition of Indefinite Integrals.

2.2 INDEFINITE INTEGRALS (DENOTED BY ∫ SIGN)

If f and F are functions of x such that $F'(x) = f(x)$ then the function F is called an *antiderivative or primitive or Integral* of $f(x)$ with respect to x. Symbolically, it is written as

$$\boxed{\int f(x)dx = F(x) + c}$$

$$\Leftrightarrow \qquad \frac{d}{dx}[F(x) + c] = f(x),$$

Where c is called the constant of integration, and $f(x)$ is called the integrand.

Note: $\int f(x)dx = F(x) + c$, represents a family of curves. Here different values of c correspond to different members of this family and these members can be obtained by moving any one of the curves parallel to itself. Furthermore, if we take intersection of a line $x = a$ with the curves, the tangents to the curves at these points of intersection are parallel. This is geometrical interpretation of indefinite integral.

Properties of Indefinite Integrals

1. $\int af(x)dx = a\int f(x)dx$ *('a' is constant).*
2. *Integral of sum is equal to sum of integrals* $\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$.
3. *If* $\int f(y)dy = F(y) + c$, *then* $\int f(ax+b)dx = \frac{1}{a}F(ax+b)+c$, $a \neq 0$.

Standard Results

1. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c\,;\ n \neq -1$
2. $\int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b| + c$
3. $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$
4. $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{\log a} + c\,,\ (a > 0)$
5. $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$
6. $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$
7. $\int \tan(ax+b)dx = \frac{1}{a}\log|\sec(ax+b)| + c$
8. $\int \cot(ax+b)dx = \frac{1}{a}\log|\sin(ax+b)| + c$
9. $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$
10. $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + c$
11. $\int \operatorname{cosec}(ax+b)\cdot\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$
12. $\int \sec(ax+b)\cdot\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + c$

13. $\int \sec x dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{\pi}{2}+\frac{x}{2}\right)\right| + c$

14. $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + c = \log\left|\tan\frac{x}{2}\right| + c$

15. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

17. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

18. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log[x+\sqrt{x^2+a^2}] + c$

19. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log[x+\sqrt{x^2-a^2}] + c$

20. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

21. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

22. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

23. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2}) + c$

24. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + c$

25. $\int e^{ax}\cdot \sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

26. $\int e^{ax}\cdot \cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

Solved Examples on Integration as the Inverse Operation of Differentiation

Example 1. Evaluate $I = 2\int e^{2x}(\cos 2x - \sin 2x)dx$.

Solution: Here $2e^{2x}(\cos 2x - \sin 2x)$ is the derivative of $e^{2x}\cos 2x$

$\Rightarrow \qquad I = e^{2x}\cos 2x + c.$ **Ans.**

Example 2. Evaluate $1 = \int \frac{2dx}{\sin^2 x \cdot \cos^2 x}$.

Solution: Transform the integrand in the following way.

$$\frac{1}{\sin^2 x\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x} = \sec^2 x + \operatorname{cosec}^2 x = \frac{d}{dx}[\tan x - \cot x]$$

Hence $\qquad 1 = 2\int(\sec^2 x + \operatorname{cosec}^2 x)dx = 2\tan x - 2\cot x + c.$ **Ans.**

Solved Examples based on Standard Results

Evaluate the following:

1. $\int x^4 dx = \frac{x^{4+1}}{4+1} + c = \frac{x^5}{5} + c$

2. $\int (3x^2 - 10x + 7)dx = 3\int x^2 dx - 10\int x dx + 7\int dx = \frac{3\cdot x^3}{3} - \frac{10\cdot x^2}{2} + 7x = x^3 - 5x^2 + 7x + c$

3. $\int \frac{dx}{(x+7)} = \log|x+7| + c$

4. $\int \frac{dx}{9x-7} = \frac{1}{9}\int \frac{dt}{t} = \frac{1}{9}\log|t| + c = \frac{1}{9}\log|9x-7| + c$ (where $t = 9x - 7$ and $dt = 9\,dx$)

5. $\int e^{5x} dx = \frac{1}{5}\int e^{5x}(5dx) = \frac{e^{5x}}{5} + c$

6. $\int a^{9x} dx = \frac{1}{9}\int a^{9x}(9dx) = \frac{a^{9x}}{9\log a} + c$

7. $\int \cos 4x dx = \frac{1}{4}\int \cos 4x \cdot (4dx) = \frac{1}{4}\sin 4x + c$

8. $\int \sin\frac{x}{3} dx = 3\int \sin\frac{x}{3}\cdot\left(\frac{1}{3}dx\right) = -3\cos\frac{x}{3} + c$

9. $\int \tan 9x dx = \frac{1}{9}\int \tan 9x \cdot 9dx = \frac{1}{9}\log|\sec 9x| + c$

10. $\int x^2 \cot x^3 dx = \frac{1}{3}\int \cot x^3 (3x^2 dx) = \frac{1}{3}\int \frac{\cos x^3}{\sin x^3}\cdot(3x^2 dx) = \frac{1}{3}\log|\sin x^3| + c$

11. $\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx = \int \frac{(\sec^2 x + \sec x \tan x)}{(\sec x + \tan x)} dx = \log|\sec x + \tan x| + c$

[On substituting $(\sec x + \tan x) = t$ we get $(\sec^2 x + \sec x \tan x)dx = dt$]

12. $\int \frac{dx}{1+\cos x} = \int \frac{(1-\cos x)}{(1-\cos^2 x)} dx = \int \frac{(1-\cos x)}{\sin^2 x} dx = \int [\operatorname{cosec}^2 x - \cot x \operatorname{cosec} x] dx$

$= -\cot x + \operatorname{cosec} x + c$

13. $\int \operatorname{cosec} x dx = \int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{\sec^2\frac{x}{2}\cdot\frac{1}{2}dx}{\tan\frac{x}{2}} = \log\left|\tan\frac{x}{2}\right| + c$

14. $\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3} + c$

15. $\int \frac{dx}{16+x^2} = \frac{1}{4}\tan^{-1}\frac{x}{4} + c$

16. $\int \frac{dx}{x\sqrt{9x^2-25}} = \int \frac{3dx}{3x\sqrt{(3x)^2-5^2}} = \frac{1}{5}\sec^{-1}\frac{3x}{5}+c$

17. $\int \frac{(x+5)}{\sqrt{7-3x-x^2}}\cdot dx$

$$= \frac{-1}{2}\int \frac{(-2x-10)dx}{\sqrt{7-4x-x^2}}$$

$$= \frac{-1}{2}\int \frac{(-2x-4)-6}{\sqrt{7-4x-x^2}}dx$$

$$= \frac{-1}{2}\int \frac{(-2x-4)}{\sqrt{7-4x-x^2}}dx + \frac{6}{2}\int \frac{dx}{\sqrt{7-4x-x^2}}$$

$$= -\sqrt{7-4x-x^2} + 3\int \frac{dx}{\sqrt{11-(x+2)^2}}$$

$$= -\sqrt{7-4x-x^2} + 3\sin^{-1}\left(\frac{x+2}{\sqrt{11}}\right)+c$$

Origin of Notation ∫
The notation of integral was introduced by German mathematician Gottfried Wilhelm Leibniz in 17th century. He adapted the integral symbol, ∫, from the letter f (archaic form of 's'), standing for summa which is Latin for 'sum' or 'total'.

18. $\int \frac{dx}{(x^2-1)} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+c$

19. $\int \frac{dx}{(1-x^2)} = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|+c$

20. $\int \frac{dx}{\sqrt{x^2-1}} = \log|x+\sqrt{x^2-1}|+c$

21. $\int \frac{dx}{\sqrt{x^2+1}} = \log|x+\sqrt{x^2+1}|+c$

22. $\int \frac{dx}{\sqrt{4x^2+25}} = \frac{1}{2}\int \frac{2dx}{\sqrt{(2x)^2+5^2}} = \frac{1}{2}\log(2x+\sqrt{4x^2+25})+c$

23. $\int \sqrt{4-x^2 dx} = \frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}+c = \frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+c$

24. $\int \sqrt{x^2-9}\, dx = \frac{x}{2}\sqrt{x^2-9}-\frac{9}{2}\log|x+\sqrt{x^2-9}|+c$

25. $\int \sqrt{7x^2+11}\, dx = \frac{1}{\sqrt{7}}\int \sqrt{7x^2+11}\cdot\sqrt{7}dx$

$$= \frac{1}{\sqrt{7}}\left[\frac{\sqrt{7}}{2}\times\sqrt{7x^2+11}+\frac{11}{2}\log\{\sqrt{7}x+\sqrt{7x^2+11}\}\right]+c$$

$$= \frac{x}{2}\sqrt{7x^2+11}+\frac{11}{2\sqrt{7}}\log(\sqrt{7}x+\sqrt{7x^2+11})+c$$

2.3 IMPORTANT TECHNIQUES/METHODS OF INTEGRATION

If the integrand is not a derivative of a known function, then the corresponding integrals cannot be found directly. In order to find the integrals in such problems, following main three rules of integration are used:

Rule 1: Integration by substitution (*i.e.*, by changing variables)

Rule 2: Integration by parts.

Rule 3: Integration by partial fractions.

Rule 1: Integration By Substitution (*i.e.*, by Changing Variables)

If $f(x)$ is a continuous differentiable function, then to evaluate integrals of the form $\int \phi(f(x))f'(x)dx$, we substitute $f(x) = t$ and $f'(x)dx = dt$

Hence $I = \int \phi(f(x))f'(x)dx = \int \phi(t)dt$,which can further be integrated easily. Likewise, in integrals of the type $\int [Q(x)]^n Q'(x)dx$ or $\int \frac{Q'(x)}{\sqrt{Q(x)}}dx$ or $\int \frac{Q'(x)}{[Q(x)]^n}dx$, we put $Q(x) = t$ and solve.

> **Note:** In method of substitution, it is important to observe the question minutely and then make a substitution for a function whose derivative also occurs in the integrand.

Examples

Example 3. Evaluate $\int \frac{\cos(\log x)}{x}dx$.

Solution: Let $\log x = t \left[\text{as derivative of } \log x \text{ is } \frac{1}{x}\right]$

Then $dt = \frac{1}{x}dx$, substituting it in given integral, we get

$$I = \int \cos t\, dt = \sin t + c$$

$$I = \sin(\log x) + c$$ **Ans.**

Example 4. Evaluate $\int \frac{(1+\log x)^3}{x}dx$.

Solution: Let $I = \int \frac{(1+\log x)^3}{x}dx$

Substitute $(1 + \log x) = z$, we get $\frac{1}{x}dx = dz$

$$\Rightarrow \quad I = \int z^3 dz = \frac{z^4}{4} + c, \text{ as } z = 1 + \log x$$

$$\Rightarrow \quad I = \frac{(1+\log x)^4}{4} + c$$ **Ans.**

Example 5. Evaluate $\int e^x \cos e^x dx$.

FUN FACT
The first documented systematic technique capable of determining integrals "the method of exhaustion" was given by Greek mathematician Eudoxus about 2400 years ago!

Solution: $\int e^x \cos e^x dx = \int \cos e^x \cdot e^x dx$.

Put $e^x = t$

Then, $I = \int \cos t dt$

$\Rightarrow$ $I = \sin t + c$

$\Rightarrow$ $I = \sin e^x + c$

Rule 2. Integration by Parts

If u and v are differentiable functions of x then,

$$\int u \cdot v dx = u \int v dx - \int [u' \cdot \int v dx] dx \qquad ...(1)$$

where $u' = \dfrac{du}{dx}$

The given integral in (1) must be separated into two parts, one part (first function) being u and the other part (second function) being v ***[for this reason it is called integration by parts].***

We usually follow the rules as below for integrating with parts:

(*i*) Choose u and v such that $\int v dx$ and $\int [u' \int v dx] dx$ are simple to integrate.

(*ii*) If in the integrand only one function is there, then we take unity 1 as the second function. For instance, in $\int \sin^{-1} x dx$, $\sin^{-1} x$ is taken as the first function (u) and 1 as the second function (v).

(*iii*) Generally, we choose first function (u) as the function which comes first in the word **ILATE**, where **I** Stands for Inverse function; **L** Stands for Logarithmic function; **A** Stands for Algebraic function; **T** Stands for Trigonometric function; **E** Stands for Exponential function. For example, in $\int x^2 \cos x dx$, x^2 is taken as the first function (u) and $\cos x$ is taken as the second function (v).

Examples:

Example 6. Evaluate $\int \sec^3 \theta d\theta$

Solution: Let $I = \int \sec\theta \cdot \sec^2 \theta d\theta$

$\Rightarrow$ $I = \sec\theta \int \sec^2 \theta d\theta - \int \tan\theta(\sec\theta \tan\theta) d\theta$

$\Rightarrow$ $I = \sec\theta \tan\theta - \int (\sec^2 \theta - 1)\sec\theta d\theta$

$\Rightarrow$ $I = \sec\theta \tan\theta - \int \sec^3 \theta d\theta + \log|\sec\theta + \tan\theta| + c_1$

$\Rightarrow$ $2I = \sec\theta \tan\theta + \log|\sec\theta + \tan\theta| + c_1$

$\Rightarrow$ $I = \dfrac{1}{2}\sec\theta \tan\theta + \dfrac{1}{2}\log|\sec\theta + \tan\theta| + c \left(c = \dfrac{c_1}{2} \right)$

Example 7. Evaluate $\int \log x dx$.

Solution: As only one function is there in the integrand we take unity (1) as the second function

$\therefore$ If $I = \int 1 \cdot \log x dx$

Then $$I = \log x\int 1dx - \int\left[\left[\frac{d}{dx}(\log x)\right]\times\int 1\cdot dx\right]dx + c$$

$\Rightarrow$ $$I = x\log x - \int\frac{1}{x}\cdot xdx + c \quad\Rightarrow\quad I = x\log x - x + c \quad \textbf{Ans.}$$

Example 8. Evaluate $\int e^x \sin x dx$.

Solution: Let $$I = \int e^x \sin x dx$$

Integrating by parts and taking sin x as first function and e^x as second function we get

$$I = \sin x\int e^x dx - \int e^x(\cos x)dx + c$$

$\Rightarrow$ $$I = e^x\sin x - \left[e^x\cos x - \int e^x(-\sin x)dx\right] + c \quad \text{(again integrating by parts)}$$

$\Rightarrow$ $$I = e^x\sin x - e^x\cos x - \int e^x\sin x dx + c \qquad \left(\text{as } I = \int e^x\sin x dx\right)$$

$\Rightarrow$ $$2I = e^x\sin x - e^x\cos x + c$$

$\Rightarrow$ $$I = \frac{1}{2}[e^x\sin x - e^x\cos x] + c$$

Rule 3: Integration By Partial Fractions

A function of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials, is called a rational fraction.

It is called proper rational fraction if the degree of $f(x)$ is less than the degree of $g(x)$, otherwise is called an improper rational fraction. An improper rational fraction can further be expressed as the sum of a polynomial and a proper rational fraction (long division process can be used for it). Every proper rational fraction can be expressed as a sum of simpler fractions, called partial fractions. Here, we restrict our study to linear factors in the denominators of partial fraction and hence the cases which arise are listed in the table 2.1, where A_1, A_2, A_3 are constants to be determined accordingly.

Table 2.1

Sr. No.	Proper Rational Fraction	Partial Fraction
1.	$\frac{px+q}{(ax+b)(cx+d)}$	$\frac{A_1}{ax+b}+\frac{A_2}{cx+d}$
2.	$\frac{px+q}{(ax+b)^2(cx+d)}$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(cx+d)}$
3.	$\frac{px^2+qx+r}{(ax+b)(cx+d)(ex+f)}$	$\frac{A_1}{ax+b}+\frac{A_2}{cx+d}+\frac{A_3}{ex+f}$
4.	$\frac{px^2+qx+r}{(ax+b)^2(cx+d)}$	$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(cx+d)}$

Recall: *A polynomial in x is a function of the form* $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ *where* a_i*'s are constants for all i,* $a_0 \neq 0$ *and n is a positive integer including zero*

Examples:

Example 9. $I = \int \frac{(3x+1)}{(x+1)(x-2)} dx$.

Solution: Here the integrand is a proper rational fraction. So simplifying it further into partial fractions, we get

$$\frac{3x+1}{(x+1)(x-2)} = \frac{A_1}{x+1} + \frac{A_2}{x-2}$$

$$\Rightarrow \quad (3x + 1) = A_1(x - 2) + A_2(x + 1) \quad ...(1)$$

Now, putting $x = 2$ in (1) we get

$$7 = 3A_2 \Rightarrow A_2 = \frac{7}{3} \text{ and putting } x = -1 \text{ gives } -2 = -3A_1 \Rightarrow A_1 = \frac{2}{3}$$

$$\therefore \quad I = \frac{2}{3}\int \frac{dx}{x+1} + \frac{7}{3}\int \frac{dx}{x-2}$$

$$\Rightarrow \quad I = \frac{2}{3}\log|x+1| + \frac{7}{3}\log|x-2| + c \quad \textbf{Ans.}$$

Example 10. Evaluate $\int \frac{(3x-4)}{(x-1)^2(x+1)} dx$.

Solution: Let $$I = \int \frac{(3x-4)}{(x-1)^2(x+1)} dx$$

Integrand is a proper rational fraction and hence simplifying it further using partial fractions we get

$$\frac{3x-4}{(x-1)^2(x+1)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+1}$$

$$\Rightarrow \quad (3x - 4) = A_1(x - 1)(x + 1) + A_2(x + 1) + A_3(x - 1)^2 \quad ...(1)$$

Putting $x = 1$ in (1) we get

$$-1 = 2A_2 \Rightarrow \boxed{A_2 = \frac{-1}{2}}$$

Again put $x = -1$, in (1) we get

$$-7 = 4A_3 \Rightarrow \boxed{A_3 = -\frac{7}{4}}$$

Now equating coefficient of constant terms on both sides of (1) we get

$$-4 = -A_1 + A_2 + A_3$$

$$A_1 = 4 + A_2 + A_3$$

$$\Rightarrow \quad A_1 = 4 - \frac{1}{2} - \frac{7}{4} \quad \left[\text{as } A_2 = -\frac{1}{2}; A_3 = \frac{-7}{4}\right]$$

$$\Rightarrow \quad \boxed{A_1 = \frac{7}{4}}$$

$$\therefore \qquad I = \frac{7}{4}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{(x-1)^2} \frac{-7}{4}\int\frac{dx}{x+1}$$

$$\Rightarrow \qquad I = \frac{7}{4}\log|x-1| + \frac{1}{2(x-1)} - \frac{7}{4}\log|x+1| + c$$

$$I = \frac{7}{4}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{2(x-1)} + c \qquad \left(\text{as } \log\frac{m}{n} = \log m - \log n\right)$$

2.4 DEFINITE INTEGRALS

The definite integral of a continuous function $f(x)$ defined on the closed interval $[a, b]$, is given by $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$; where $F(x)$ is the anti-derivative of f, 'a' is called the lower limit of the integral and 'b' is called the upper limit of the integral. The definite integral has a *unique value*.

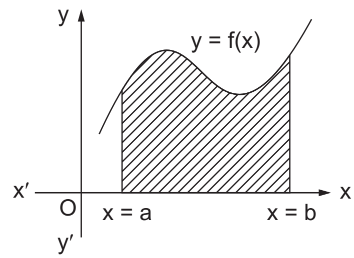


Fig. 2.2

Geometrically, the definite integral $\int_a^b f(x)dx$ represents the algebraic area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. (see Fig. 2.2)

Remarks:

1. $\int_a^b f(x)dx = 0 \Rightarrow$ the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b)
2. $\int_a^a f(x)dx = 0$ (*i.e.*, when lower limit = upper limit)

Example 11. Evaluate:

(*i*) $\int_0^2 x^3 dx$ (*ii*) $\int_0^{\pi/4} \sin^4 2t \cos 2t\,dt$

Solution:

(*i*) Let $I = \int_0^2 x^3 dx$

As we know that, $\int x^3 dx = \frac{x^4}{4} = F(x)$

$$\therefore \qquad I = F(2) - F(0) = \frac{16}{4} - 0 = 4 \qquad \textbf{Ans.}$$

(*ii*) Let $$I = \int_0^{\pi/4} \sin^4 2t \cos 2t\,dt$$

Then $$\int \sin^4 2t \cos 2t\,dt = \frac{1}{2}\int z^4 \cdot dz$$ [Put sin $2t = z \Rightarrow 2\cos 2t\,dt = dz$]

$$= \frac{z^5}{10} = \frac{(\sin 2t)^5}{10} = F(t)$$

Then $$I = F\left(\frac{\pi}{4}\right) - F(0) = \frac{1}{10}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$\Rightarrow$ $$I = \frac{1}{10}$$ **Ans.**

Some Common Properties of Definite Integrals

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$ *provided f is same.*
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, *where c may lie inside or outside the interval [a, b]. This property is to be used when f is piecewise continuous in (a, b).*
4. $\int_{-a}^a f(x)dx = \int_0^a [f(x)+f(-x)]dx = \begin{cases} 0; & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx; & \text{if } f(x) \text{ is an even function} \end{cases}$
5. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, *in particular* $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
6. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx = \begin{cases} 2\int_0^a f(x)dx; & \text{if } f(2a-x) = f(x) \\ 0; & \text{if } f(2a-x) = -f(x) \end{cases}$
7. $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$, *($n \in I$); where 'T' is the period of the function i.e., $f(T+x) = f(x)$*

Note that: $\int_x^{T+x} f(t)dt$ will be independent of x and equal to $\int_0^T f(t)dt$

Examples:

Example 12. Evaluate $\int_0^2 f(x)dx$, if $f(x) = \begin{cases} x^3 & 0 < x < 1 \\ 2x+1 & 1 \le x \le 2 \end{cases}$

Solution:
$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$$
$$= \int_0^1 x^3 dx + \int_1^2 (2x+1)dx$$
$$= \left[\frac{x^4}{4}\right]_0^1 + [x^2+x]_1^2 = \frac{1}{4} + [4+2-2] = \frac{1}{4} + 4 = \frac{17}{4} \quad \textbf{Ans.}$$

Interesting Analogy
Differentiation : Derivative :: Integration : Integral

Example 13. Evaluate $I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$.

Solution: Let
$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \cdot dx \qquad ...(1)$$
$$\Rightarrow \qquad I = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}} dx$$
$$\left(\text{Using property of definite integral } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$
$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \cdot dx \qquad ...(2)$$

From (1) and (2) we get
$$2I = \int_0^{\pi/2} \frac{(\sqrt{\sin x} + \sqrt{\cos x})}{(\sqrt{\sin x} + \sqrt{\cos x})} \cdot dx \quad \Rightarrow \quad I = \frac{1}{2}\int_0^{\pi/2} dx$$
$$\Rightarrow \qquad I = \frac{1}{2}[x]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{2}\right] \quad \Rightarrow \quad I = \frac{\pi}{4} \quad \textbf{Ans.}$$

2.5 USE OF WALLI'S INTEGRAL FORMULA $\left(\int_0^{\pi/2} \sin^m x \cos^n x dx\right)$

This formula is attributed to the English Mathematician John Wallis (1616-1703). Its proof is based on reduction formulae. We will study following two cases:

Case I. $\int_0^{\pi/2} \sin^m x \cos^n x dx$. The value of this definite integral with trigonometric integrand can be found easily using Walli's integral formula as given below:

If m, n are both positive integers, then

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \int_0^{\pi/2} \sin^n x \cos^m x dx$$

$$\begin{cases} = \dfrac{(m-1)(m-3)...(1 \text{ or } 2)\cdot(n-1)(n-3)...(1 \text{ or } 2)}{(m+n)(m+n-2)...(1 \text{ or } 2)}\cdot\dfrac{\pi}{2}, & \text{if } m \text{ and } n \text{ both are even integer} \\ = \dfrac{(m-1)(m-3)...(1 \text{ or } 2)\cdot(n-1)(n-3)...(1 \text{ or } 2)}{(m+n)(m+n-2)...(1 \text{ or } 2)}\cdot 1, & \text{Otherwise} \end{cases}$$

Case II. If n is a positive integer, then

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \begin{cases} \dfrac{(n-1)(n-3)(n-5)...3\cdot 1}{n(n-2)(n-4)...4\cdot 2}\cdot\dfrac{\pi}{2}; & \text{if } n \text{ is even} \\ \dfrac{(n-1)(n-3)(n-5)...4\cdot 2}{n(n-2)(n-4)...5\cdot 3}\cdot 1; & \text{if } n \text{ is odd} \end{cases}$$

This is special case of Walli's integral formula.

Example 14. Evaluate $\int_0^{\pi/2} \sin^6 x \cos^2 x dx$.

Solution: Using Walli's integral formula we get

$$\int_0^{\pi/2} \sin^6 x \cos^2 x dx = \frac{(6-1)(6-3)(6-5)\cdot(2-1)}{(6+2)(6+2-2)(6+2-4)(6+2-6)}\cdot\frac{\pi}{2}$$

(as both 6 and 2 are even integers)

$$= \frac{5\cdot 3\cdot 1\cdot 1}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{15}{384}\cdot\frac{\pi}{2} = \frac{15}{768}\cdot\pi$$

Examples: Evaluate the following:

15. $I = \int_{-\pi/2}^{\pi/2} \sin^4 x \cos^6 x dx \Rightarrow I = 2\int_0^{\pi/2} \sin^4 x \cos^6 x dx$ (as the integrand is an even function)

$$\Rightarrow \quad I = \frac{2\cdot(3\cdot 1)(5\cdot 3\cdot 1)}{10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi}{256}$$ **Ans.**

16. $I = \int_0^{\pi/2} \cos^7 x dx$. Using Wallis formula, as the exponent is odd, so we get

$$I = \frac{6\times 4\times 2}{7\times 5\times 3} = \frac{16}{35}$$ **Ans.**

17. $I = \int_0^{\pi/2} \sin^8 x dx$. Using Walli's formula and as the exponent is even in I we get,

$$I = \frac{7\times5\times3\times1}{8\times6\times4\times2}\times\frac{\pi}{2} \quad \Rightarrow \quad I = \frac{35\pi}{256}$$ **Ans.**

2.6 APPLICATIONS OF INTEGRATION

Integrals are applied to many fields like physics, economics etc. apart from being used in calculating volume, surface area, arc length, average value of functions etc. Traditional methods of geometry help us in calculating area, volume etc. of simple figures. But these conventional methods cannot be used for calculating the areas, volume etc. enclosed by curves. Here is where, the concept of integral calculus comes in to our rescue.

In this section, we would learn to deal with simple problems related to evaluation of area bounded by a curve and axes. Apart from this we would learn to calculate the volume of a solid formed by revolution of an area about axes (simple problems only).

I. Area Bounded by a Curve and Axes

We learnt that definite integrals help us in finding area. When we find area between a curve and axes the following two cases arise:

Case I. When we are finding the area A bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ as shown in fig. 2.3.

In this case we have to find out the area enclosed in $ABDCA$. This area is considered as being made up of many very-thin vertical strips /rectangles. We consider one arbitrary strip of height y and width dx. For this strip area $dA = ydx$. (here $y = f(x)$). This area is called an *elementary area*. We sum up all such elementary areas by taking integration from $x = a$ to $x = b$ (in $ABDCA$) and get the total area bounded by the curve $y = f(x)$, x-axis and the lines $x = a$, $x = b$ as below:

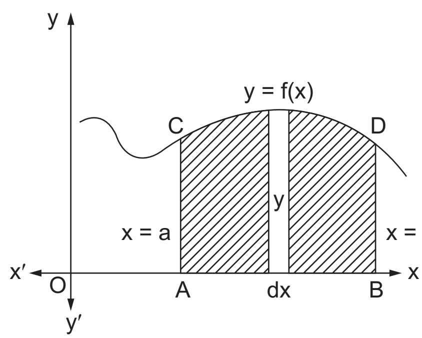


Fig. 2.3

$$A = \int_a^b dA = \int_a^b ydx = \int_a^b f(x)dx \ \ i.e., \ \boxed{A = \int_a^b f(x)dx}$$

Case II: When we are finding the area A bounded by the curve $x = g(y)$, y -axis and the lines $y = c$, $y = d$, as shown in (Fig. 2.4). In this case we have to find out area enclosed in $PQRSP$. Here, we consider area $PQRSP$ as made up of very thin horizontal strips/rectangles of length x and width dy.

$\therefore$ Area $\boxed{A = \int_c^d g(y)dy}$

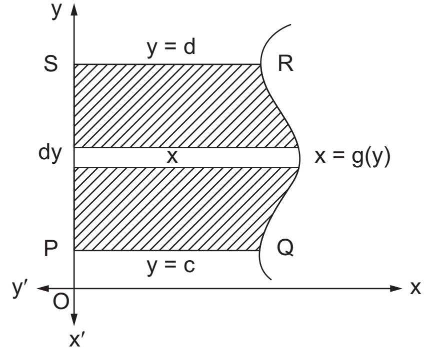


Fig. 2.4

Remember:

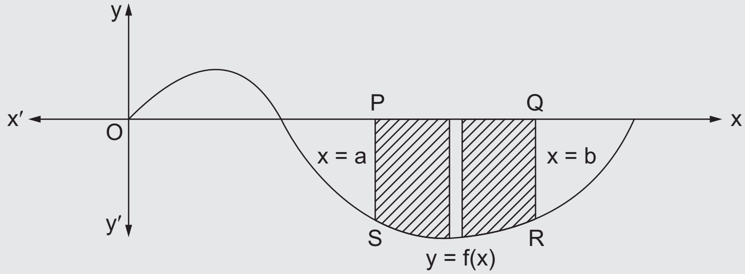


Fig. 2.5

1. It might happen at times that value of area A under consideration comes out to be negative. But, in such cases we consider only the numerical value of the area, *i.e.*, take its absolute value $|A|$.

   For example: Consider the curve given in (Fig. 2.5). In this curve, area A under consideration is $PQRSP$, which is below x-axis and negative as $y = f(x) < 0$. In such cases, we consider its absolute value *i.e.*, $|\mathrm{A}| = \left|\int_a^b f(x)dx\right|$.

2. Equation of X-axis is given by $y = 0$

   Equation of Y-axis is given by $x = 0$

Example 18. Evaluate the area enclosed by one quadrant of the circle $x^2 + y^2 = 1$. Hence find the total area enclosed by the given circle.

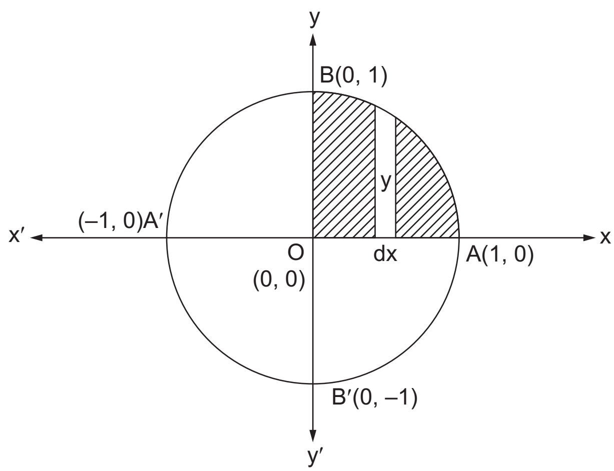


Fig. 2.6

Solution: The circle under consideration is $x^2 + y^2 = 1$ ...(1)

As circle is symmetrical along both the axes and points of intersection of (1) with x-axis are $A(1, 0)$ and $A'(-1, 0)$ and with y-axis are $B(0, 1)$ and $B'(0, -1)$ we get Fig. 2.6. As area of all quadrants is equal we find the area enclosed by quadrant $OABO$.

i.e., $$Q_1 = \int_0^1 y\,dx$$

By taking vertical strip

$$Q_1 = \int_0^1 \sqrt{1-x^2}\,dx$$

(as $y = \pm\sqrt{1-x^2}$ taking the positive sign as $OABO$ lies in first quadrant from (1)

$$\Rightarrow \quad Q_1 = \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\frac{x}{1}\right]_0^1$$

$$\Rightarrow \quad Q_1 = \left[\frac{1}{2}\sin^{-1}1\right] \qquad \left(\text{as } \sin^{-1}1 = \frac{\pi}{2}\right)$$

$$\Rightarrow \quad Q_1 = \frac{\pi}{4}$$ **Ans.**

Again, area A enclosed by the circle = $4 \times Q_1$

$$\Rightarrow \quad A = 4\times\frac{\pi}{4}$$

$$\Rightarrow \quad A = \pi \text{ sq. units}$$ **Ans.**

Alternate solution: *You can solve this question using horizontal strips too in positive quadrant. Try!*

Example 19. Evaluate the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

Solution: Equation of ellipse in

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad ...(1)$$

The points of intersection of (1) with x-axis and y-axis are $A(a, 0)$; $A'(-a, 0)$ and $B(0, b)$; $B'(0, -b)$ respectively.

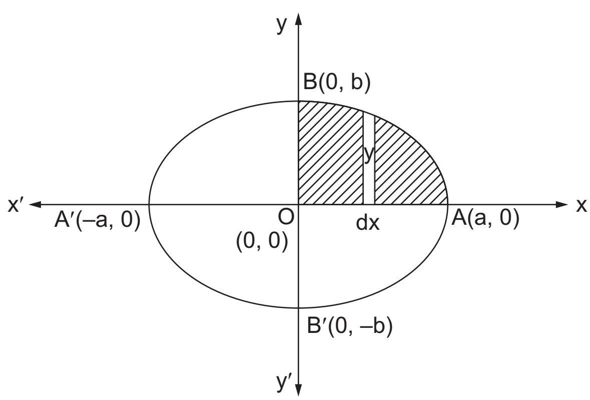


Fig. 2.7

Again ellipse is symmetrical about both the axes.

So take vertical strips in first quadrant as shown in the Fig. 2.7 we get,

Area of ellipse $= 4 \times$ (Area of first quadrant $OABO$)

$\Rightarrow$ $4\times$ (area of the region $OABO$ in first quadrant bounded by the ellipse, x-axis and lines $x = 0$, $x = a$)

$$= 4\times\int_0^a y\,dx = 4\times\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

[as $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \Rightarrow y \pm \frac{b}{a}\sqrt{a^2-x^2}$ but taking positive sign $y = \frac{b}{a}\sqrt{a^2-x^2}$, as $OABO$ lies is first quadrant]

We get, $$A = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$\Rightarrow \quad A = \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$\Rightarrow \quad A = \frac{4b}{a}\left[\frac{a^2}{2}\sin^{-1}\frac{a}{a}\right] = \frac{4b}{a}\left[\frac{a^2}{2}\sin^{-1}1\right]$$

$$A = \cancel{2}ba\cdot\frac{\pi}{\cancel{2}} = \pi\,ab \qquad \left[\text{as } \sin^{-1}1 = \frac{-\pi}{2}\right]$$

i.e., $A = \pi\,ab$ sq. units **Ans.**

II. Volume of a Solid formed by Revolution of an Area about Axes (Using Disk Method)

One of the important and simplest applications of integration is to find the volumes of solid formed by revolution. Following two cases arise:

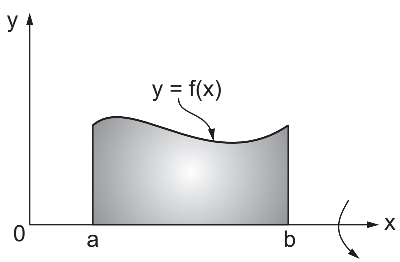


Fig. 2.8

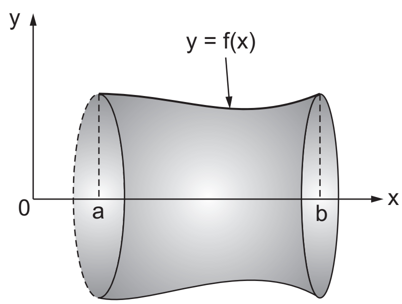


Fig. 2.9

Case I: Revolution about X-axis: The volume of a solid formed by revolution of an area about X-axis, bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the X-axis is (Fig. 2.8 and Fig. 2.9)

$$\boxed{V = \int_a^b \pi y^2\,dx}$$

Case II: **Revolution about *Y*-axis:** The volume of a solid formed by revolution of an area about *Y*-axis, bounded by the curve $x = f(y)$, the lines $y = c$, $y = d$ and the *Y*-axis is

$$\boxed{V = \int_c^d \pi x^2 dy} \qquad \text{(Fig. 2.10 and Fig. 2.11)}$$

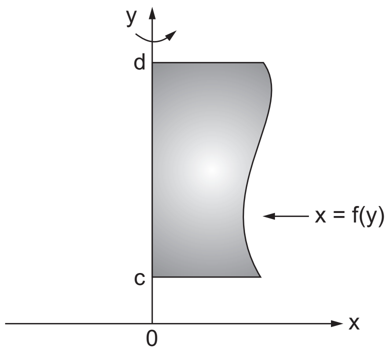


Fig. 2.10

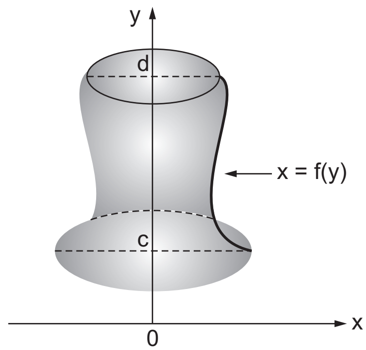


Fig. 2.11

Example 20. Let $y = f(x) = x^2$ be the curve given on the interval [0, 2]. Find the volume of solid generated by revolving the region between the curve in the given interval around *X*-axis.

Solution: We know that volume of solid of revolution around *X*-axis in $[a, b]$ is given by (for $y = f(x)$)

$$V = \int_a^b \pi y^2 dx \;\; i.e., V = \int_0^2 \pi (x^2)^2 dx \Rightarrow V = \pi \int_0^2 x^4 dx \Rightarrow V = \pi \left| \frac{x^5}{5} \right|_0^2 \Rightarrow V = \frac{32\pi}{5} \;\; \textbf{Ans.}$$

Example 21. Let $y = f(x) = x^2$ be given on the interval [0, 2]. Find the volume of solid generated by revolving the region between the curves (in the given interval [0, 2]) around *Y*-axis.

Solution: As here we have to find the revolution around *y*-axis,

$$\therefore \qquad V = \int_a^b \pi x^2 dy \;\; \text{formula will be used. Here } y = x^2 \Rightarrow x^2 = y \text{ on } [0, 2]$$

$$\therefore \qquad V = \pi \int_0^2 y\,dy \;\; \Rightarrow \;\; V = \pi \left[\frac{y^2}{2} \right]_0^2 \;\; \Rightarrow \;\; V = 2\pi \quad \textbf{Ans.}$$

Example 22. Evaluate the volume of the solid generated by the revolution of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its minor axis.

Solution: Minor axis of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; (b < a) \text{ is the } Y\text{-axis}$$

$\therefore$ We use the formula for revolution around *Y*-axis *i.e.*,

$$V = \int_a^b \pi x^2 dy \;\; \text{on interval } [a, b]$$

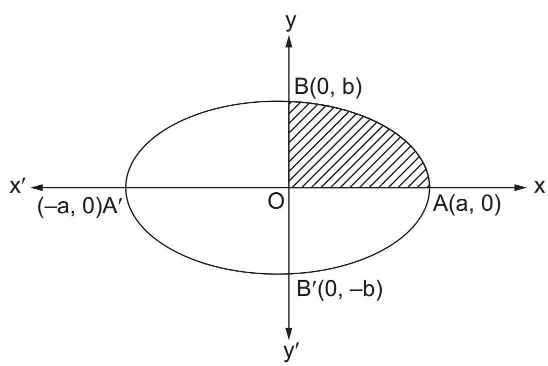


Fig. 2.12

Now, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad x^2 = \dfrac{a^2}{b^2}(b^2-y^2)$

$\therefore$ Volume of the ellipse about its minor axis (Y-axis)

= volume generated by revolution of arc $B'AB$ about Y-axis (Fig. 2.12)

= twice the volume generated by revolution of the arc BA about Y-axis (as ellipse is symmetrical about X-axis). But the arc BA varies from $y = 0$ to $y = b$

$\therefore$ Required volume $= 2\displaystyle\int_0^b \pi x^2 dy$

$$\Rightarrow \quad V = 2\pi\int_0^b \frac{a^2}{b^2}(b^2-y^2)dy \quad \Rightarrow \quad V = 2\pi\frac{a^2}{b^2}\int_0^b (b^2-y^2)dy$$

$$\Rightarrow \quad V = \frac{2\pi a^2}{b^2}\left[b^2 y-\frac{y^3}{3}\right]_0^b \quad \Rightarrow \quad V = \frac{2\pi a^2}{b^2}\left[b^3-\frac{b^3}{3}\right] \quad \Rightarrow \quad \boxed{V=\frac{4\pi a^2 b}{3}} \qquad \textbf{Ans.}$$

Miscellaneous Applications of Integral Calculus

It finds applications in most of the professions/fields and hence touches our lives each day in numerous ways. Some are listed below:

- Dams (especially like Bhakra Dam in India) are engineering marvels. When the reservoir behind the dams are full, the dams with stands a great deal of force. This hydrostatic force exerted is calculated with the help of definite integrals when the reservoir is full. We can also examine how changing water levels of reservoir affect this force.
- Integration is used to calculate the exact length of power cable needed to connect miles away substations (electrical engineering).
- Widely used in various concepts of physics like kinematics, thermodynamics, electromagnetism, centre of mass, centre of gravity, moment of inertia, predict position of planets, velocity, force, work done etc.
- Finding area, arc length, volume of three-dimensional solids etc.
- For constructing curved shaped structures and to measure weight of such structures, architects use integration.

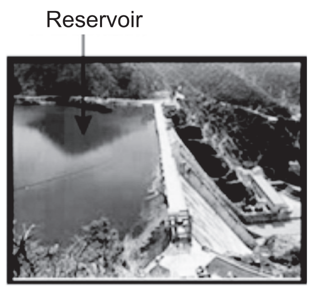


Fig. 2.13: Bhakra Dam [Source: Ministry of Water Resources, GOI retrieved from old.cwc.gov.in]

Video Resource References [Source: NCERT]

Properties of definite Integrals

Applications of Integrals: Introduction

Application of Integrals: Area under the Curve

Application of Integrals: Area Between Two Curves

Application of Integrals: Problems

CASE STUDY

In a memory test, the rate at which the students memorized mathematics symbols is given by $F'(t) = \frac{t}{10} - \frac{3t^2}{1000}$ where $F(t)$ is the number of symbols memorized in t minutes.

Answer the following questions based on above data:

1. Evaluate $F(t)$ if it is known that $F(0) = 0$
2. How many symbols can a student memorize in 10 minutes (approximately)?

Check Out!!!!

Download MATLAB (https://www.mathworks.com/downloads/) (Source: MathWorks) free trial version after unit 5 is taught and check whether you can evaluate the value of $\int x^n dx$ and $\int_0^1 x^7 dx$.

Math Works

SUMMARY

1. $\boxed{\int f(x)dx = F(x) + c}$, where f and F are function of x such that $F'(x) = f(x)$.
2. Important techniques of integration:

   2.1. **Integration by substitution:** To evaluate integrals of the form $\int \phi(f(x)) f'(x)dx$ we substitute $f(x) = t$ and $f'(x)\,dx = dt$.

   2.2. **Integration by parts:** If u and v are differentiable functions of x then,

$$\int u \cdot v dx = u\int v dx - \int [u' \cdot \int v dx] dx \text{ where, } u' = \frac{du}{dx}$$

[Generally, we choose first function (u) as the function which comes first in the word **ILATE**, where **I** Stands for Inverse function; **L** Stands for Logarithmic function; **A** Stands for Algebraic function; **T** Stands for Trigonometric function; **E** Stands for Exponential function].

2.3 **Integration by partial fractions:** A function of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials, is called a rational fraction. Every proper rational fraction can be expressed as a sum of simpler fractions, called partial fractions and then can be integrated accordingly.

3. The definite integral is given by $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$; where $F(x)$ is the anti-derivative of f, 'a' is called the lower limit of the integral and 'b' is called the upper limit of the integral.
4. Walli's integral formula is, if m, n are both positive integers, then

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \int_0^{\pi/2} \sin^n x \cos^m x dx$$

$$\begin{cases} = \dfrac{(m-1)(m-3)\ldots(1 \text{ or } 2)\cdot(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}\cdot\dfrac{\pi}{2}, & \text{if } m \text{ and } n \text{ both are even integer} \\ = \dfrac{(m-1)(m-3)\ldots(1 \text{ or } 2)\cdot(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}\cdot 1, & \text{Otherwise} \end{cases}$$

5. If n is a positive integer, (special case of Walli's integral formula) then

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\ldots 3\cdot 1}{n(n-2)(n-4)\ldots 4\cdot 2}\cdot\dfrac{\pi}{2} & \text{if } n \text{ is even} \\ \dfrac{(n-1)(n-3)(n-5)\ldots 4\cdot 2}{n(n-2)(n-4)\ldots 5\cdot 3}\cdot 1 & \text{if } n \text{ is odd} \end{cases}$$

6. Area A bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is $\boxed{A = \int_a^b f(x)dx}$
7. Area A bounded by the curve $x = g(y)$, y-axis and the lines $y = c$, $y = d$ is $\boxed{A = \int_c^d g(y)dy}$
8. The volume of a solid formed by revolution of an area about X-axis, bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the X-axis is $V = \int_a^b \pi y^2 dx$
9. The volume of a solid formed by revolution of an area about Y-axis, bounded by the curve $x = f(y)$, the lines $y = c$, $y = d$ and the Y-axis is $V = \int_c^d \pi x^2 dy$.

Subjective Questions

Q.1. Solve: $\int \sqrt{\dfrac{1-x}{1+x}}dx$. [**Hint.** Put $x = \cos\theta$] [**Ans.** $\sqrt{1-x^2} - \cos^{-1} x + c$]

Q.2. Evaluate (*a*) $\int \tan^{-1} x dx$ (*b*) $\int \sin(\log x) dx$

$$\left[\textbf{Ans. } (a)\ x \tan^{-1} x - \frac{1}{2}\log(1+x^2) + c, (b)\ \frac{x}{2}[\sin(\log x) - \cos(\log x)] + c\right]$$

Q.3. Prove that $\displaystyle\int_0^{\pi/4} \log(1+\tan x)dx = \frac{\pi}{8}\log 2$

Objective Questions

Q.1. If $f'(x) = \dfrac{1}{(2\sin x + 3\cos x)^2}$, then $f(x)$ is equal to _______. [**Hint.** Put ($2 \tan x + 3 = t$)]

$$\left[\textbf{Ans. } \frac{-1}{2(2\tan x + 3)} + c\right]$$

Q.2. $\displaystyle\int \frac{dx}{\sqrt{x + x\sqrt{x}}}$ = _______. [**Hint.** Put $\sqrt{x} = t$] [**Ans.** $4\sqrt{1+\sqrt{x}} + c$]

Q.3. $\int 4dx$ = _______. [**Ans.** $4x + c$]

Q.4. $\int 5e^x dx$ = _______. [**Ans.** $5e^x + c$]

Match the following:

Q.5. The antiderivative of -

Table 2.2

Column I		Column II	
(*A*)	$(n+1)x^n$	(*i*)	$\log\lvert\sin x\rvert + c$
(*B*)	$\cos x$	(*ii*)	$x^{n+1} + c$
(*C*)	$\cot x$	(*iii*)	$\tan x + c$
(*D*)	$\sec^2 x$	(*iv*)	$(\sin x) + c$

[**Ans.** $A \to (ii)$; $B \to (iv)$; $C \to (i)$; $D \to (iii)$]

Mini Project

Form groups of five students. Each group has to make a table of 50 anti-derivatives and respective derivatives. Tally how many similar anti derivatives a group has when compared to other groups (make a table for this too). Show it to your teacher.

Activity

Choose any random part of a land which is easily accessible to you (but not polygonal in shape). Try to find out its area using definite integrals. Thereafter find its area with the help of measuring tape. Compare and analyze your answers. Show it to your teacher.

Know More

In calculating many integrals we had to use the method of integration by parts several times in succession. The result could be obtained more rapidly by using the generalized formula for integration by parts

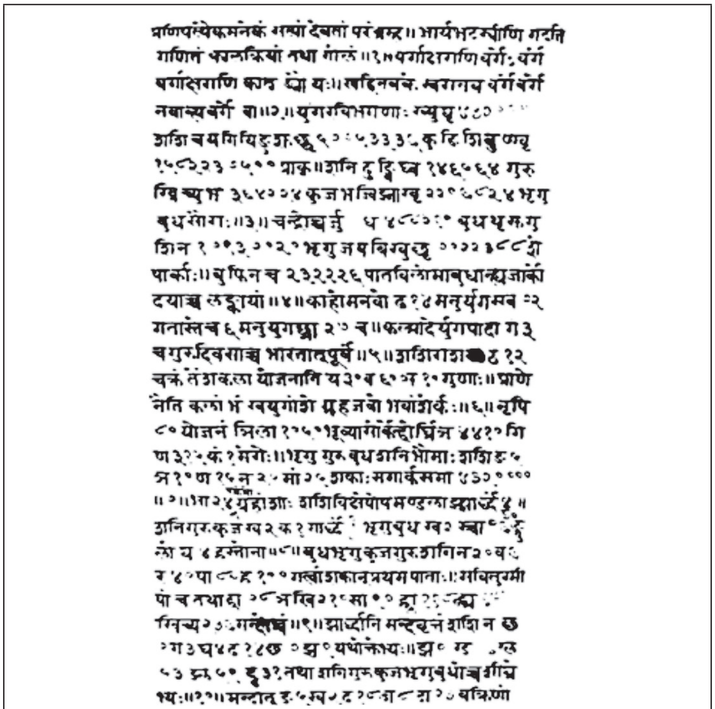
$$\int u(x)v(x)dx = u(x)v_1(x) - u'(x)v_2(x) + u''(x)v_3(x) - \ldots + (-1)^{n-1}u^{n-1}(x)v_n(x) - (-1)^{n-1}\int u^n(x)v_n(x)dx$$

where $v_1(x) = \int v(x)dx$, $v_2(x) = \int v_1(x)dx$ ..., $v_n(x) = \int v_{n-1}(x)dx$

Here, we assume that all derivatives and integrals appearing in this formula exist. The use of the generalized formula for integration by parts is especially useful when calculating $\int P_n(x)Q(x)dx$, where $P_n(x)$ is polynomial of degree n and the factor $Q(x)$ is such that it can be integrated successively $(n + 1)$ times.

References and Suggested Readings

- NCERT, Mathematics Text Book for Class XII Part II
- Narayan Shanti (1942) Integral Calculus, S. Chand & Co.
- Ayres Frank (1981), Calculus, Schaum's outline series.



An excerpt of ten verses Dashgeetikapaad'
from **"Aryabhateeya"** written by *Aryabhatta*
(Source: Muley Gunakar (1992), Sansaar ke Mahan Ganitagya, Raajkamal Prakashan)

3 Co-ordinate Geometry

UNIT SPECIFICS

This unit discusses the topics - equation of straight line in various standard forms (without proof), inter section of two straight lines, angle between two lines, parallel and perpendicular lines, perpendicular distance formula along with solved examples. General equation of a circle, its characteristics; definition of conics (Parabola, Ellipse, Hyperbola), their standard equations have also been explained in length.

RATIONALE

The way every person on earth has a home address, similarly every point has an address known as its coordinates (which are unique). The study of geometry using a coordinate system is known as co-ordinate geometry. It is not just a stepping stone to higher maths but also a key to understanding the visual aspects of various structures. It is an important branch for present day explorations of outer space and for behavioral study of atomic particles. Again, coordinate geometry concept is used in address system of New York city which makes it easier to find an address as compared to the traditional ones. Hence, coordinate geometry is very important as it is multifariously used, like- in physics, in GPS, in maps, in computer graphics and in various other fields in different disguises.

Pre-requisites

- Basics of geometry like lines, angles etc.
- Distance formula.
- Knowledge related to solving an equation.
- Knowledge of abscissa/ordinate.

Unit 3 Integral Calculus	Unit Outcomes (UO) The students will learn to -
U3-O1	Find the equation of straight line in various forms; use appropriate formulas to find angle between two lines and perpendicular distance between two lines.
U3-O2	Determine the intersection points between two straight lines;understand the concept of parallel and perpendicular lines.
U3-O3	Find the equation of circle and relate its characteristics accordingly.
U3-O4	Understand the concept of conics with reference to right circular cone.
U3-O5	Solve problems based on parabola, ellipse and hyperbola when their foci, directories or vertices are given.

CO-UO Mapping

Unit 2 Outcomes	Expected mapping with course outcomes (1 - Weak correlation; 2 - Medium correlation; 3 - Strong correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U3-O1			3	1	2
U3-O2			3	1	2
U3-O3			3	1	2
U3-O4			3	1	2
U3-O5			3	1	2

3.1 CONCEPT OF COORDINATE GEOMETRY

The mathematical formulations of algebra and geometry are designated as coordinate geometry. For the first time is the history of mathematics the use of algebra and geometry simultaneously was carried out by the then great French Mathematician **René Descartes**.

The algebraic combination of analysis and geometry is referred to as analytical geometry.

Origin of Coordinate Geometry
French mathematician René Descartes first carried out a systematic study of geometry by the use of algebra in his book 'La Géométrie' (published in 1637). Modern coordinate (analytic) geometry is called 'Cartesian' after him.

Cartesian Co-ordinates System

In order to have the concept of cartesian coordinates system we consider the geometrical space. Consider an X-Y plane, in which there are two dimensions. One is called x-axis and the other is called y-axis. Since there are two axes (perpendicular) in this geometrical space that's why it is called two dimensional rectangular coordinate system. The horizontal line is labelled as X-axis and the vertical line as Y-axis. At this stage we make it clear that X-axis is symbolic as independent variable and Y-axis is symbolic as dependent variable. In Fig. 3.1 in the horizontal, X-axis is taken and in the vertical, Y-axis is taken. It is to be noted in the Cartesian system, the coordinates of a point are written as (x, y), in which, first place is specifically meant for X-axis and second place is meant for Y-axis. $O(0, 0)$ is the intersection point between X-axis and Y-axis.

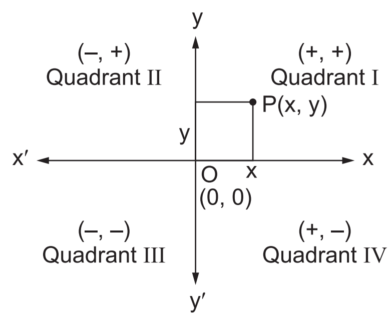


Fig. 3.1

3.2 STRAIGHT LINE

Consider two vertices (points) on a geometrical space say P and Q. The shortest distance between the two vertices P and Q is called straight line. Any straight line can be viewed as a curve but without curvature.

If P and Q are any two vertices denoted as $P(x_1, y_1)$ and $Q(x_2, y_2)$, then distance between

P and Q is, $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

If the two vertices are lying on each other than $PQ = 0$

Example 1. The number of vertices on X-axis, which are at a distance $a(a < 4)$ from another vertex $Q(3, 4)$ is

(1) 2 (2) 3 (3) 4 (4) not defined

Solution: Let a vertex on x-axis be $(x_1, 0)$ then its distance D from the vertex (3, 4) is $a = \sqrt{(x_1-3)^2+(0-4)^2}$

$$\Rightarrow \quad a = \sqrt{(x_1-3)^2+16} \Rightarrow a^2 = (x_1-3)^2 + 16 \Rightarrow (x_1-3)^2 = a^2 - 16$$

$$\Rightarrow \quad x_1 - 3 = \pm\sqrt{a^2-16} \Rightarrow x_1 = 3\pm\sqrt{a^2-16} \Rightarrow x_1 = 3\pm\sqrt{a^2-16}$$

As $a < 4$, therefore, x_1 is not defined. **Ans.** (4)

Remark: A relation in x and y which is satisfied by co-ordinates of every point on a line is known as equation of straight line. It is given by $ax + by + c = 0$. (a and $b \neq 0$)

Equation of Vertical Lines

(1) Equation of Y-axis is $x = 0$

(2) Equation of a line parallel to Y-axis at distance of a units is $x = a$ or $x = -a$. (Fig. 3.2)

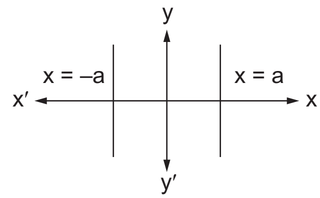


Fig. 3.2

Equation of Horizontal Line

(1) Equation of X-axis is $y = 0$

(2) Equation of a line parallel to X-axis at a distance of 'b' units is $y = b$ or $y = -b$ (Fig. 3.3)

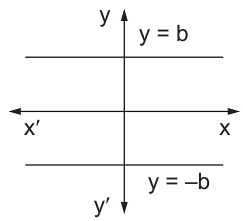
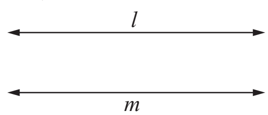


Fig. 3.3

Parallel lines: Two lines are said to be parallel if they do not have any intersection vertex this implies that the two lines are disjoint.



(l and m are parallel lines)

Note: Equation of line parallel to $ax + by + c = 0$ is $ax + by + \lambda = 0$ (λ-parameter)

Perpendicular lines: Two lines are said to be perpendicular lines if they intersect at 90°angle.

Equation of a line perpendicular to line $ax + by + c = 0$ is $bx - ay + \lambda_1 = 0$ (λ_1 = parameter). **Ex.** Co-ordinate axes are perpendicular lines. (Fig. 3.4)

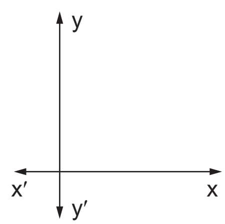


Fig. 3.4

Coincident lines: Two lines are said to be coincident on each other if one line overlaps the other lines.

It is to be noted that when two lines (parallel) are moving in the same direction, they will meet at a vertex (point) which is called vertex at infinity. But geometrically it is not possible to construct.

Example 2. Prove that the lines

(*a*) $2x + 5y + c = 0$; $4x + 10y + 2c_1 = 0$ are parallel to each other.

(*b*) $3x + 4y + 7 = 0$; $4x - 3y + 1 = 0$ are perpendicular to each other

Solution: (*a*) Let
$$2x + 5y + c = 0 \quad ...(1)$$
$$4x + 10y + 2c_1 = 0 \quad ...(2)$$

$$(1) \Rightarrow \quad y = -\frac{2}{5}x - \frac{c}{5}$$

$$(2) \Rightarrow \quad y = -\frac{4}{10}x - \frac{2c_1}{10}$$

$$\Rightarrow \quad y = -\frac{2x}{5}x - \frac{c_1}{5}$$

$$m_2 = \frac{-2}{5} = m_1 \quad \therefore \text{(1) and (2) are parallel lines.} \quad \textbf{Ans.}$$

(*b*) Given lines are

$$3x + 4y + 7 = 0 \text{ and } 4x - 3y + 1 = 0$$

i.e., $$y = \frac{-3}{4}x - \frac{7}{4} \text{ and } y = \frac{4}{3}x + \frac{1}{3}$$

i.e., $$m_1 = -\frac{3}{4} \text{ and } m_2 = \frac{4}{3}$$

$\therefore$ $$m_1 m_2 = \left(\frac{-3}{4}\right) \times \left(\frac{4}{3}\right) = -1.$$ Hence lines are perpendicular to each other. **Ans.**

3.2.1 Slope of a Line

Suppose a line PQ makes an angle $\phi (0° \le \phi < 180°)$ with the right side directions of X-axis, then slope of this line will be designated by $\tan \phi$ and is denoted by symbol m i.e., $m = \tan \phi$. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points such that, $x_1 \neq x_2$, then slope of the line $PQ = \dfrac{y_2 - y_1}{x_2 - x_1}$

Remark:

(*i*) If $\phi = 90°$, then slope m does not exist and line is disjoint to Y-axis.
(*ii*) If $\phi = 0$, then $m = 0$ and the line is disjoint to X-axis.
(*iii*) (*a*) If lines are disjoint then $m_1 = m_2$ and vice versa.
(*b*) If lines are perpendicular to each other, then $m_1 \cdot m_2 = -1$ and vice versa.

3.2.2 Equation of Straight Line in Various Standard Forms

(*i*) **Slope intercept form:** Suppose m be the slope of a line and c its intercept on Y-axis. Then the equations of such straight line is given as

$$y = mx + c \quad ...(1)$$

where y and x are dependent and independent variables respectively. If in this equation we put $c = 0$, equation (1) reduces to $y = mx$.

(*ii*) **Point slope from:** If m be the slope of a straight line which passes through a point (x_1, y_1), then its equation is written as $y - y_1 = m(x - x_1)$

(*iii*) **Intercept form:** Suppose c_1 and c_2 be the two intercepts obtained by a straight line on the X-axis and Y-axis, then its equation is written as $\frac{x}{c_1}+\frac{y}{c_2} = 1$.

Note:

(*i*) Distance of intercept of line between the coordinate axis = $\sqrt{c_1^2+c_2^2}$

(*ii*) Area of triangle $OPQ = \frac{1}{2}PO \cdot QO = \left|\frac{1}{2}c_1 \cdot c_2\right|$

(*iii*) **Two point form:** Equations of a line going through two points (x_1, y_1) and (x_2, y_2) is expressed as

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

where $|x|$ denotes determinant of x.

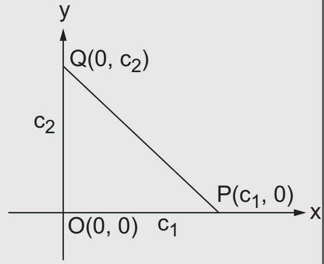


Fig. 3.5

Example 3. Find the equation of line passing through (0, 0) and (2, 2).

Solution: Using the formula

$$y - y_1 = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)\text{, we get } y - 0 = \left(\frac{2-0}{2-0}\right)(x-0)$$

$$\Rightarrow \quad 2y = 2x \quad \Rightarrow \quad y = x \quad \textbf{Ans.}$$

(*iv*) **Normal form:** If p is the distance of perpendicular on a straight line from the origin, and θ is the angle between the positive X-axis and this perpendicular, then the equation of this straight line is expressed as

$x \cos\theta + y \sin\theta = p$ (p is always greater then 0) where $0 \le \theta \le 2\pi$

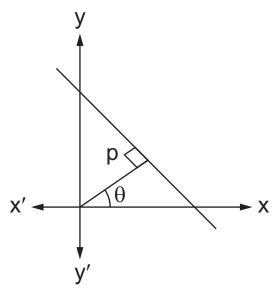


Fig. 3.6

(*v*) **General form:** Consider two variables x and y and three constants a, b, c. The linear or first degree equation in variables x and y is represented as

$$ax + by + c = 0 \qquad \text{...(1)}$$

This is the equation of a straight line. The algebraic structure given in (1) is the general form of straight line. Now we derive the following from (1)

(*i*) Slope of the line = $\frac{-a}{b} = -\frac{\text{coefficient of } x}{\text{coefficient of } y}$

(*ii*) Intercept by this line on x-axis = $-\frac{c}{a}$ and also the intercept by this line on y-axis = $-\frac{c}{b}$.

(*iii*) In order to change the general form of straight line into the normal form we take c to the R.H.S of (1) and convert it to positive, then divide the entire equation by $\sqrt{a^2+b^2}$.

3.2.3 Angle between Two Lines

(*a*) Angle ϕ between two straight lines (say)

$$y = a_1x + c_1 \text{ and } y = a_2x + c_2, \text{ is}$$

$$\tan\phi = \pm\left(\frac{a_1 - a_2}{1 + a_1a_2}\right)$$

It is to be noted that-

(*i*) There are two angles generated between two straight lines but usually the acute angle is taken as the angle between two lines. Hence the value of ϕ can be determined on considering only the positive value of $\tan\phi$.

(*ii*) Let a_1, a_2, a_3 be the slopes of three straight lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$, where $a_1 > a_2 > a_3$ then the interior angles of any triangle found by these formulae are given by (ΔEFG)

$$\tan E = \frac{a_1 - a_2}{1 + a_1a_2},\ \tan F = \frac{a_2 - a_3}{1 + a_2a_3},\ \tan G = \frac{a_3 - a_1}{1 + a_3a_1}$$

(*b*) Consider the equation of two straight lines-

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

then these straight lines are

(*i*) Parallel $\Leftrightarrow$ $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

(*ii*) Perpendicular $\Leftrightarrow$ $a_1a_2 + b_1b_2 = 0$ $\qquad (\because a_1a_2 = -1)$

(*iii*) Coincident $\Leftrightarrow$ $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Remark: *Straight line making a given angle with a line* - Equations of the straight lines going through a point whose coordinates are (x_1, y_1) and forming an angle (say) α, with the straight line whose linear equation is $y = mx + c$ is written as

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Example 4. Which of the following is correct value if $3x + 4y - 5 = 0$ and $2x + ky + 6 = 0$ are two equations of two perpendicular lines

(1) $-\dfrac{3}{2}$ (2) $\dfrac{3}{2}$ (3) 3 (4) 4

Solution: Slope of the 1st line = $-\dfrac{\text{coefficient of } x}{\text{coefficient of } y}$

Consider equation $3x + 4y - 5 = 0$, we have

$$m_1 = \frac{-3}{4}.$$

From equation $2x + ky + 6 = 0$, we have

$$m_2 = \frac{-2}{k}$$

If $m_1 m_2 = -1$, then the two straight lines are perpendicular and we have

$$m_1 = \frac{-3}{4},\ m_2 = \frac{-2}{k}$$

$$\therefore \qquad m_1 m_2 = -1 \Rightarrow -\frac{-3}{4}\times\frac{-2}{k} = -1$$

$$\frac{3}{2k} = -1 \Rightarrow 3 = -2k \Rightarrow k = \frac{-3}{2} \quad \textbf{Ans. } (1)$$

Example 5. Find the equation to the sides of an isosceles right angled triangle, the equations of whose hypotenuse is $5x + 6y = 6$ and the opposite vertex is the point $P(3, 3)$.

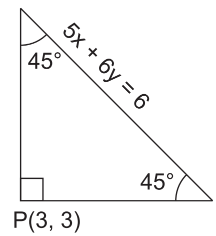


Fig. 3.7

Solution: Obtain the equation of the straight lines passing through the given point (3, 3) and making equal angles of 45° with the given straight line $5x + 6y = 6$ or $5x + 6y - 6 = 0$. Slope of the straight line $5x + 6y - 6 = 0$ is given by

$$m_1 = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y} = -\frac{5}{6}$$

$$\tan 45° = \pm\frac{m - m_1}{1 + mm_1}$$

or $$1 = \pm\frac{m - m_1}{1 + mm_1} = +\frac{m + \frac{5}{6}}{1 + m\left(\frac{-5}{6}\right)} = +\frac{6m + \frac{5}{6}}{6 - 5\frac{m}{6}} \qquad (\because \tan 45° = 1)$$

$$\Rightarrow \qquad 1 = +\frac{6m+5}{6-5m} \Rightarrow 6 - 5m = 6m + 5 \quad \text{or} \quad m_A = \frac{1}{11},$$

Now consider $$1 = -\frac{6m+5}{6-5m} \Rightarrow 6 - 5m = -6m - 5$$

or $$6 + 5 = -6m + 5m \quad \text{or} \quad 11 = -m \quad \text{or} \quad m_B = -11$$

Hence the required equations of two straight lines are

$$y - 3 = m_A(x - 3)$$

$$\Rightarrow \qquad y - 3 = \frac{1}{11}(x - 3) \Rightarrow x - 11y + 30 = 0$$

and $$y - 3 = m_B(x - 3)$$

$$\Rightarrow \qquad y - 3 = -11(x - 3) \Rightarrow 11x + y - 36 = 0$$

3.2.4 Distance of Perpendicular from a Point on a Line

Let us consider a straight line whose equation is given as

$ax + by + c = 0$ then, distance of perpendicular from a point $P(x_1, y_1)$ is expressed as $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

In particular, the distance of the perpendicular from the origin (0, 0) lying on the line

$$ax + by + c = 0 \text{ is } p = \frac{|c|}{\sqrt{a^2 + b^2}}, \text{ (since } x = 0, y = 0)$$

Example 6. Verify whether any line can be drawn through the point (4, –6) so that its distance from (–2, 3) will be equal to 10.

Solution: Let the equations on straight line through point $P(4, -6)$ so that its distance from (–2, 3) is equal to 10. We know that the equation is constructed on taking the point $P(4, -6)$ with slope of m is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

We have $\quad y + 6 = m(x - 4)$

$\Rightarrow \quad y + 6 = mx - 4m \quad \Rightarrow \quad mx - y - 4m - 6 = 0$

then $\quad \dfrac{|m(-2) - 3 - 4m - 6|}{\sqrt{m^2+1}} = 10$

$$\frac{|-6m-9|}{\sqrt{m^2+1}} = 10 \quad \Rightarrow \quad \frac{6m+9}{\sqrt{m^2+1}} = 10 \quad \text{or} \quad 6m + 9 = 10\sqrt{m^2+1}$$

Squaring both the sides, we have

$$6m + 9 = 10\sqrt{m^2+1}$$

$$(6m + 9)^2 = 100(m^2 + 1)$$

$$\Rightarrow \quad 36m^2 + 108m + 81 = 100m^2 + 100$$

$$\Rightarrow \quad 64m^2 - 108m + 19 = 0$$

Now the discriminant $b^2 - 4ac$ is

$$(108)^2 - 4(64)(19) = 11664 - 4864 = 6800 > 0 \text{ which is positive.}$$

Hence such line is possible.

3.2.5 Distance between Two Parallel Lines

(*i*) The distance between two disjoint (parallel) lines

$$ax + by + c_1 = 0 \quad \text{and} \quad ax + by + c_2 = 0 \text{ is determined as } D = \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$$

It is to be noted that in case of parallel (disjoint) lines the coefficients of x and y in both equations must be same.

(*ii*) The area of parallelogram= $\dfrac{q_1q_2}{\sin\theta}$, where q_1 and q_2 are length between two pairs of opposite edge and ϕ is the angle between any two adjacent edges.

It is to be noted that area of the parallelogram bounded by the four straight lines say,

$$y = m_1x + c_1 \quad \ldots(i) \qquad y = m_1x + c_2 \quad \ldots(ii)$$

$$y = m_2x + d_1 \quad \ldots(iii) \qquad y = m_2x + d_2 \quad \ldots(iv)$$

is given by $\quad z = \left|\dfrac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$

Example 7. The given equations of 3 lines are

$$x + 2y + 3 = 0 \quad \ldots(1)$$

$$x + 2y - 8 = 0 \quad \ldots(2)$$

$$3x - y - 5 = 0 \quad ...(3)$$

Form 3 sides of two squares. Find the equation of remaining sides of these squares.

Solution: We make use of the formula to find the distance between the two parallel lines as

$$D = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

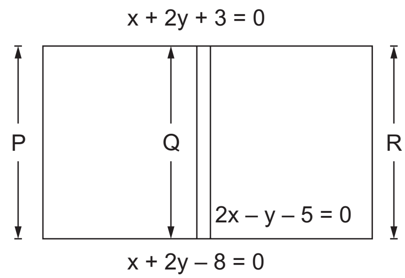


Fig. 3.8

Where a is the coefficient of x and b is the coefficient of y. c_1, c_2 are constant terms. Consider equations (1) and (2) to find the value of D, we have

$$D = \frac{|8+3|}{\sqrt{5}} = \frac{11}{\sqrt{5}}$$

The equations of sides P and R are of the form

$$2x - y + k = 0$$

Since distance between sides P and Q = distance between sides Q and R

$$\frac{|k-(-5)|}{\sqrt{5}} = \frac{11}{\sqrt{5}}$$

$\Rightarrow$ $$\frac{k+5}{\sqrt{5}} = \frac{11}{\sqrt{5}} \text{ or } k + 5 = 11 \text{ or } k = 11 - 5 = 6$$

and $$\frac{k+5}{\sqrt{5}} = -\frac{11}{\sqrt{5}} \text{ or } k + 5 = -11 \text{ or } k = -11 - 5 = -16$$

In order to obtain the fourth sides of the two squares are (*i*) $2x - y + 6 = 0$ and (*ii*) $2x - y - 16 = 0$.

Circle

3.3 CONCEPT OF CIRCLE

A circle is a closed two -dimensional circular figure in which the set of all the points in the plane is equidistant from a given fixed point called the "centre" of the circle and the constant distance is called the radius of the circle.

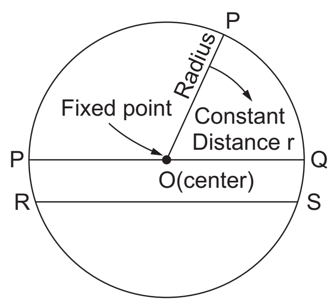


Fig. 3.9

Diameter of a circle (PQ) = $2 \times r = 2r$.

Circumference of a circle = $C = 2\pi r$. It is the length of boundary of the circle.

The area of a circle is area bounded by the circumference and computed using the following formula, that is

$$\text{Area } (A) = \pi r^2$$

where π is a constant quantity and is equal to $\frac{22}{7}$ or $\pi = 3.14$ and r is the radius of the circle.

The line joining any two points on the circumference is called a chord. If the chord passes through centre then it is called diameter. In Fig. (3.9) RS = chord, PQ = diameter, O = centre

Note: A circle is a special case of an ellipse (conic section).

3.3.1 General Equation of Circle

Let us write the mathematical formulation of the general circle equations as

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(1)$$

where x, y are variables, g, f and c are constants and centre is $(-g, -f)$ *i.e.*,

$$\left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right) \text{ and radius } r = \sqrt{g^2 + f^2 - c}\,.$$

> **Note:** If radius r is real then the circle is known as a real circle; if radius $r = 0$ then the circle is a point/vertex circle; if radius r is imaginary then the circle is also imaginary.

Example 8. Find the centre and radius of the circles

(*a*) $2x^2 + 2y^2 - 6x - 8y + 2 = 0$ (*b*) $x^2 + y^2 + 2x \sin\theta + 2y \cos\theta - 10 = 0$

(*c*) $2x^2 + \lambda xy + 2y^2 + (\lambda - 2)x + 4y - 6 = 0$ for some λ.

Solution:

(*a*) We reconstruct the given equations as

$$x^2 + y^2 - \frac{6}{2}x - \frac{8}{2}y + \frac{2}{2} = 0 \;\Rightarrow\; x^2 + y^2 - 3x - 4y + 1 = 0 \qquad ...(i)$$

We derive from equation (1) the value of g, f and c, as $g = \frac{-3}{2}$, $f = \frac{-4}{2}$, $c = 1$

Hence centre is $\left(\frac{3}{2}, 2\right)$ and the radius is

$$\sqrt{g^2 + f^2 - 1} = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2 - 1} = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$$

Therefore radius $r = \frac{\sqrt{21}}{2}$

(*b*) Consider the equation $x^2 + y^2 + 2x \sin\theta + 2y \cos\theta - 10 = 0$

Centre of this circle is $(-\sin\theta, -\cos\theta)$

Therefore, radius $= \sqrt{(-\sin\theta)^2 + (\cos\theta)^2 + 10} = \sqrt{\sin^2\theta + \cos^2\theta + 10} = \sqrt{1 + 10}$

$(\because \sin^2\theta \cos^2\theta = 1))$

$\therefore$ $r = 3.3$

(*c*) Consider the given equations of the circle as

$$2x^2 + \lambda xy + 2y^2 + (\lambda - 2)x + 4y - 4 = 0 \qquad ...(ii)$$

We rewrite the equation after dividing the given equation by the coefficient of x^2 *i.e.*, 2 we get

$$\frac{2x^2}{2} + \frac{\lambda xy}{2} + \frac{2y^2}{2} + \frac{(\lambda - 2)\cdot x}{2} + \frac{4y}{2} - \frac{4}{2} = 0$$

or $$x^2 + \frac{\lambda}{2}\cdot xy + y^2 + \frac{(\lambda - 2)}{2}\cdot x + 2y - 2 = 0 \qquad ...(iii)$$

In the general equations of a circle, there is no term of xy but in our equation (*ii*), there is a term of xy whose coefficient is $\frac{\lambda}{2}$,

that is $\therefore$ $\frac{\lambda}{2} = 0 \Rightarrow \lambda = 0$

So equation (*iii*) reduces to

$$x^2 + y^2 - x + 2y - 2 = 0 \qquad ...(iv)$$

Hence centre is $\left(\frac{1}{2}, -1\right)$ *i.e.*, $\left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$

$$\text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{1}{2}\right)^2 + 1^2 + 2} = \frac{\sqrt{13}}{2}$$

3.3.2 Characteristics of a Circle

(*i*) The diameter of a circle is the longest chord of the circle and divides circle into two equal halves.

(*ii*) The perpendicular drawn on any given chord of a circle from the centre of the circle bisect the chord. (Perpendicular bisector)

(*iii*) For a given length of perimeter the circle is the shape with largest area.

(*iv*) The circles are said to be congruent if they have equal radii.

(*v*) The angle in a semi-circle is always 90°. (angle *ACB* in Fig. 3.10)

(*vi*) All circles are similar, regardless of the measure of radii or diameters.

(*vii*) Equal chords of a circle subtend equal angles at the centre.

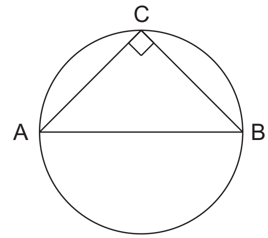


Fig. 3.10

Note:

1. The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ has three constants, hence to get the general equation of the circle at least three conditions should be known, this implies that a unique circle passes through three non collinear points.
2. A general second degree equation in x and y with variables *h*, *g*, *f*, *c* as constants is written as $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$. It represents a circle if -
   - Coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$
   - Coefficient of $xy = 0$ implies that $h = 0$
   - $(g^2 + f^2 - c) \geq 0$ (for a real circle).

Example 9. A pizza has a diameter of 10 inches. What is the radius of the pizza?

Solution: As diameter = 2 × radius

$$\therefore \qquad \text{radius} = \frac{10}{2} = 5 \text{ inches} \quad \textbf{Ans.}$$

3.3.3 Find Equation of Circle Given

I. Centre and radius.

II. Three points lying on it.

III. Coordinates of end points of diameter.

I. Centre and Radius

If (l, m) is the centre and r is the radius of the circle then its equation is given as

$$(x - l)^2 + (y - m)^2 = r^2 \qquad ...(1)$$

Particular cases:

(*i*) If centre is origin (0, 0) and radius is 'r', then equation of a circle given in (1) reduces to $x^2 + y^2 = r^2$

(*ii*) If radius of circle is zero, then equation of circle reduces to $(x - l)^2 + (y - m)^2 = 0$ (zero/point circle)

(*iii*) When circle is drawn in such a way that it touches x-axis then equation of circle is

$$(x - l)^2 + (y - m)^2 = m^2 \qquad (r = m) \qquad \text{(Fig. 3.11)}$$

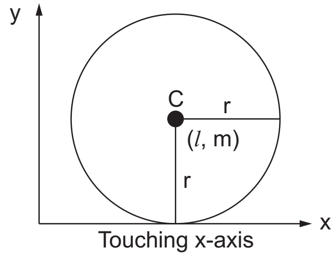


Fig. 3.11

(*iv*) When circle is drawn in such a way that one part of it is touching the y-axis, then the equations takes the form

$$(x - l)^2 + (y - m)^2 = l^2 \qquad \text{(Fig. 3.12)}$$

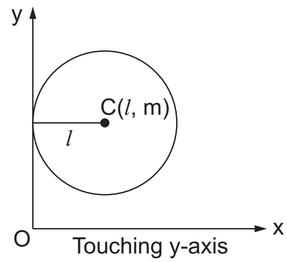


Fig. 3.12

(*v*) When circle touches both axes (x-axis and y-axis) then equation of the drawn circle is

$$(x - l)^2 + (y - l)^2 = l^2 \qquad \text{(Fig. 3.13)}$$

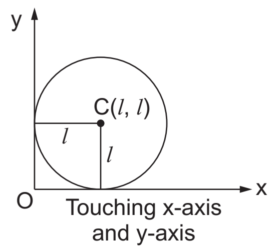


Fig. 3.13

(*vi*) When circle passes through the origin and centre of the circle is at a fixed point $C(l, m)$, then radius $(l^2 + m^2)^{1/2} = r$ and intercept cut on x-axis, is

$OP = 2l$ and intercept cut on y-axis, is $OQ = 2m$ and hence,equation of circle is $(x - l)^2 + (y - m)^2 = l^2 + m^2$ or $x^2 + y^2 - 2lx - 2my = 0$.

It is very important to note at this stage of study that centre of the circle may lie in any quadrant, hence for general cases use ± sign before l and m, *i.e.*, the coefficients of l and m are either +1 or −1. (Fig. 3.14)

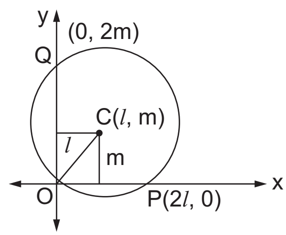


Fig. 3.14

Note: When expanded form of circle is given we can find out its centre and radius.

Rule for finding the centre point and radius of a circle. Consider the general equation of circle,

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

(*i*) Write the equation in such an algebraic form so that the coefficients of x^2 and y^2 are each equation to unity.

If it is not the case, then divide both sides of general equation by the coefficient of either x^2 or y^2, which are the same (equal) in the case of circle.

(*ii*) Coordinates of the centre of the circle are $\left(-\frac{1}{2} \text{ coefficient } x,\ -\frac{1}{2} \text{ coefficient of } y\right)$

(*iii*) Radius of the circle = $\left[\left(\frac{1}{2} \text{ coefficient of } x\right)^2 + \left(\frac{1}{2} \text{ coefficient of } y\right)^2 - \text{constant term}\right]^{1/2}$

Nature of the circle derived on the basis of the values of radius of a circle (see section 3.4)

Example 10. Find the equations of that diameter of the circle $x^2 + y^2 - 4x + 2y - 10 = 0$, which passes through the origin.

Solution: The coordinates of the centre C of the given circle is given by $\left[-\frac{1}{2}(-4), -\frac{1}{2}(2)\right]$ or (2, –1)

Therefore, required diameter is the straight line joining the origin (0, 0) and the centre $C(2, -1)$ and hence the required equation is obtained using the formula,

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

In this case $y_1 = 0, y_2 = -1, x_1 = 0, x_2 = 2$, we have

$$y - 0 = \frac{-1-0}{2-0}(x-0) \text{ or } y = \frac{-1}{2}x \text{ or } 2y = -x \text{ or } x + 2y = 0 \textbf{ Ans.}$$

Example 11. Find the equation of the circle through (3, –8) and (–6, 6) and having the centre on

$$2x - y = 7.$$

Solution: Consider the general equation of the circle as

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(i)$$

If this passes through (3, –8), then we have

$$(3)^2 + (-8)^2 + 2g(3) + 2f(-8) + c = 0 \quad \text{or} \quad 9 + 64 + 6g - 16f + c = 0$$

or $$73 + 6g - 16f + c = 0 \qquad ...(ii)$$

If it passes through (–6, 6), we have from (*i*)

$$(-6)^2 + (6)^2 + 2g(-6) + 2f(6) + c = 0$$

$$36 + 36 - 12g + 12f + c = 0$$

$$-12g + 12f + c = -72 \qquad ...(iii)$$

As the centre point $(-g, -f)$ lies on $2x - y = 7$, so, we have

$$2(-g) - (-f) = 7 \quad \text{or} \quad -2g + f = 7 \qquad ...(iv)$$

From (*ii*) and (*iv*), we get

$$-13f + c = -52 \qquad ..(v)$$

From (*iii*) and (*iv*), we get

$$6f + c = -114 \qquad ...(vi)$$

Subtracting (*vi*) from (*v*), we get

$$\begin{array}{rcl} -13f + \cancel{c} & = & -52 \\ -6f + \cancel{c} & = & -114 \\ - \quad\quad + & & \\ \hline -19f & = & 62 \end{array}$$

$$\Rightarrow \qquad f = \frac{-62}{19} = -3.3$$

Therefore from (*v*), we get

$$c = -52 - 13 \times 3.3 = -52 - 42.9 = -94.9$$

From (*iv*), we get

$$-2g - 3.3 = 7 \Rightarrow -2g = 7 + 3.3 = 10.3$$

$$\therefore \qquad g = \frac{-10.3}{2} = -5.1$$

Now we have, $f = -3.3, g = -5.1, c = -94.9$

Hence the required equation is

$$x^2 + y^2 + 2(-5.1)x + 2(-3.3)y - 94.9 = 0$$

$$x^2 + y^2 - 10.2x - 6.6y - 94.9 = 0 \quad \textbf{Ans.}$$

II. Equation of a Circle through Three Given Points

In order to derive the equation of a circle through three given vertices, consider three points say $L(x_1, y_1)$, $M(x_2, y_2)$ and $N(x_3, y_3)$ and let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...}(i)$$

Since the three points $L(x_1, y_1)$, $M(x_2, y_2)$ and $N(x_3, y_3)$ all lie on (i), so we have,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{...}(ii)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \text{...}(iii)$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \quad \text{...}(iv)$$

Eliminating, g, f and c from (i), (ii), (iii), and (iv) we get the required equations, which is the following structure

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Corollary: If the four points (x_r, y_r), where $r = 1, 2, 3, 4$ lie on a circle, then

$$\begin{vmatrix} x^2 + y^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0$$

Example 12. Find the equation of the circle which passes through the points $L(0, 1)$, $M(1, 0)$ and $N(3, 2)$. Also find the value of radius and coordinates of the centre.

Solution: Suppose the general equation of a circle is given as

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...}(i)$$

If it is going through the point $L(0, 1)$ then

$$(0)^2 + (1)^2 + 2g(0) + 2f(1) + c = 0 \quad \text{or} \quad 1 + 2f + c = 0 \quad \text{or} \quad 2f + c + 1 = 0 \quad \text{...}(ii)$$

If it passes through the point $M(1, 0)$ then, we have

$$1^2 + 0^2 + 2g(1) + 2f(0) + c = 0 \quad \text{or} \quad 1 + 2g + c = 0 \quad \text{or} \quad 2g + c + 1 = 0 \quad \text{...}(iii)$$

If it passes through the point $N(3, 2)$, then, we get

$$3^2 + 2^2 + 2g(3) + 2f(2) + c = 0$$

$$9 + 4 + 6g + 4f + c = 0$$

$$6g + 4f + c + 13 = 0 \quad \text{...}(iv)$$

Let us consider equation (ii) and (iii) as

$$2f + c + 1 = 0$$

$$2g + c + 1 = 0$$

From these two, we get

$$2f = 2g \quad \text{or} \quad f = g \qquad ...(v)$$

From (*iv*) and (*v*), we get

$$6g + 4g + c + 13 = 0 \quad \text{or} \quad 10g + c + 13 = 0 \qquad ...(vi)$$

Subtracting (*iii*) from (*vi*), we get

$$8g = -12 \quad \text{or} \quad g = -\frac{12}{8} = -1.5 \qquad ...(vii)$$

From (*v*), we get $f = g = -1.5$ and from (*ii*), we get

$$2(-1.5) + c + 1 = 0 \quad \text{or} \quad -3.0 + c + 1 = 0 \quad \text{or} \quad c = 3.0 - 1 = 2.0$$

Now, we have $f = g = -1.5$, $c = 2.0$, hence the required equation of the circle is obtained on putting $f = g = -1.5$, $c = 2.0$ in (*i*), we get

$$x^2 + y^2 - 2(1.5)x - 2(1.5)y + 2.0 = 0$$
$$x^2 + y^2 - 3x - 3y + 2 = 0$$

Now we have to determine the coordinates of centre c, obtained as $(-g, -f)$ or (1.5, 1.5)

$$\text{And the radius} = \sqrt{g^2 + f^2 - c} = \sqrt{(1.5)^2 + (1.5)^2 - 2} = \sqrt{2.25 + 2.25 - 2}$$
$$= \sqrt{4.50 - 2} = \sqrt{2.50} = 1.6$$

III. Equation of Circle in Diameter Form

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are the end points of the diameter of the circle and $P(x, y)$ is the point on the circumference other than A and B on the circle, then from geometrical aspect, we know that $\angle APB = 90°$ this implies that (slope of PA) × (slope of PB) = –1.

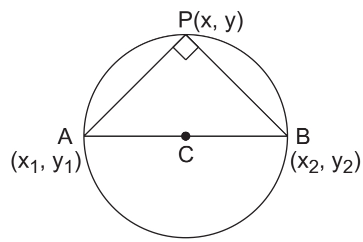


Fig. 3.15

$$\therefore \quad \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

$$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Remark: This equation is a circle having least radius passing through (x_1, y_1) and (x_2, y_2).

3.4 CONIC SECTIONS

Conic sections (or simply conics) are well defined curves obtained by the intersection of the surface of a cone with a plane. There are basically three types of conics

(*i*) Parabola (*ii*) Hyperbola (*iii*) Ellipse (Circle is a special case of ellipse)

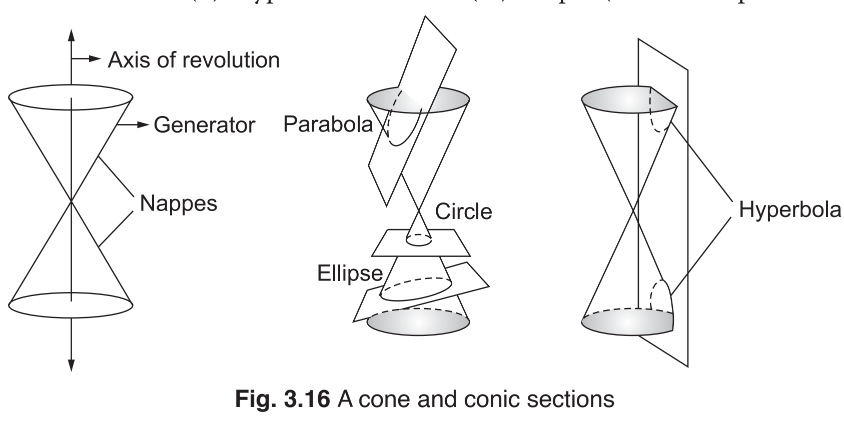


Fig. 3.16 A cone and conic sections

S.No.	When intersecting plane -	Then conics obtained is -
(i)	is parallel to the generating line	Parabola
(ii)	is parallel to the axis of revolution	Hyperbola
(iii)	intersects a nappe at an angle to the axis (other than 90°)	Ellipse
(iv)	is perpendicular (90°) to the axis of revolution	Circle

A conic section or conics is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line and this fixed point does not lie on a fixed line. The fixed point is called the focus. The constant ratio is known as the eccentricity (denoted by e). The fixed straight line is called the directrix. The line passing through its focus and perpendicular to directrix is called the axis of revolution. And point of intersection of conic with its axis of revolution is called a vertex.

General Equation of a Conic

The general equation of a conic with focus (s, t) and directrix $lx + my + n = 0$ is given as

$$(l^2 + m^2)[(x - s)^2 + (y - t)^2] = e^2(lx + my + n)$$

$$\Rightarrow \quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

If $e > 1$, the conic is called hyperbola.

If $e = 1$, the conic is called parabola.

If $e < 1$, the conic is called ellipse.

3.4.1 Parabola

A parabola is a geometrical structure which is the locus of a point, which moves in a geometrical plane such that its distance from a fixed point is always equal to its distance from a fixed straight line (*i.e.*, directrix). Standard equation of a parabola is

$$y^2 = 4ax \qquad ...(1)$$

where x and y are variables and a is constant.

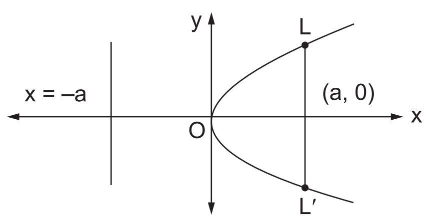


Fig. 3.17

From the equation (1) of parabola, we extract the following:

(*i*) Vertex is (0, 0)

(*ii*) Focus is $(a, 0)$

(*iii*) Axis is $y = 0$

(*iv*) Directrix is $x + a = 0$

Some Important Terms

(*a*) **Focal distance:** The length of a point on the parabola from its focus is called the focal distance of the point.

(*b*) **Focal chord:** A chord of the parabola which moves through its focus is called a focal chord.

(*c*) **Latus rectum:** The chord passes through focus and perpendicular to the directrix called axis of the parabola is called the latus rectum.

For parabola $y^2 = 4ax$, we derive the following terms:

(*i*) Length of the latus rectum = $4a$.

(*ii*) Length of the semi latus rectum = $2a$.

(*iii*) Ends of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$.

3.4.2 Hyperbola

Hyperbola is derived from conic when eccentricity $e > 1$.

Definition: The locus of a point which moves such that its length from a fixed point (focus) is e times its length from a fixed straight line (called the directrix) is defined as hyperbola. Hence hyperbola is characterized by $e > 1$.

Now we express the hyperbolic structure in mathematical formulation as follows:

The coordinate of the focus S are $(ae, 0)$ and the equation of directrix is $x = a/e$.

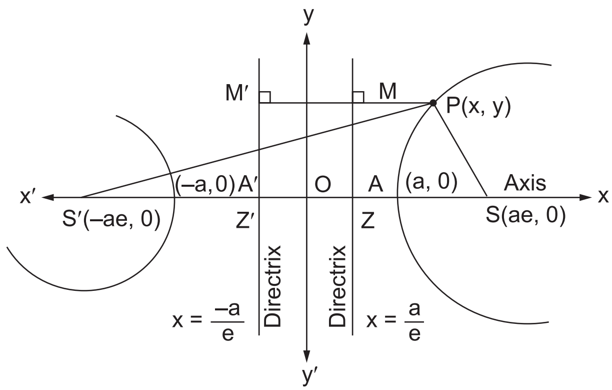


Fig. 3.18

Let $P(x, y)$ be any point on the hyperbola, then we have $SP/PM = e$ or $SP^2 = e^2 \times PM^2$

$$\text{or} \quad (x - ae)^2 + (y - 0)^2 = e^2[x - (a/e)]^2$$

$$\text{or} \quad (x - ae)^2 + y^2 = (ex - a)^2$$

$$\text{or} \quad (x^2)(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(e^2 - 1)$, since in our case $e > 1$, therefore $b^2 > 0$. Hence finally the general equation of the hyperbola is

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \quad \text{(Standard equation of hyperbola)}$$

where x and y are variables, a and b are constants and $b^2 = a^2(e^2 - 1)$. Considering the standard equation of the hyperbola, we derive the following:

(*i*) Since only even powers of x and y occur in this equation, so hyperbola is symmetrical about both the axes.

(*ii*) The hyperbola does not cut y-axis in real points whereas it cut x-axis at points $(a, 0)$ and $(-a, 0)$.

(*iii*) For the relations $-a \le x \le a$, the value of y is imaginary, that is the curve does not exist in the section $x = -a$ to $x = a$.

(*iv*) As x increases, y also increases *i.e.*, the curve extends to infinity.

(*v*) If c is length of foci from centre then $c^2 = a^2 + b^2$ for hyperbola and $e = \dfrac{c}{a}$

Some Important Definitions

Foci and Directrices: Since the curve is symmetrical about y-axis, therefore there exist another focus S' at point $(-ae, 0)$. Thus it is found that in the geometrical structure of hyperbola, there are two foci $S(ae, 0)$ and $S'(-ae, 0)$. Corresponding to these foci, there are two directrices whose equations are $x = a/e$ and $x = -a/e$.

Centre: Any chord of the hyperbola through O, the mid-point of AA' will be bisected at O and therefore O is called the centre of hyperbola (Fig. 3.18).

We also explain the hyperbola as the set of all points (x, y) such that the difference of the lengths from (x, y) to foci is constant. The standard from of an equation of a hyperbola centered at the origin with vertices $(\pm a, 0)$and co-vertices $(0, \pm b)$is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

3.4.3 Ellipse

Standard equation of ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ and $b^2 = a^2(1 - e^2) \Rightarrow a^2 - b^2 = a^2e^2$ where e = eccentricity $(0 < e < 1)$.

$$\text{Foci } F \equiv (ae, 0) \text{ and } F' \equiv (-ae, 0)$$

> **FUN FACT**
> The curves (viz. circles, ellipses, parabolas, hyperbolas) are known as *conics* because they can be obtained from a right circular cone.

(1) Vertices

$$A' \equiv (-a, 0) \text{ and } A \equiv (a, 0)$$

(2) Equation of directrices

$$x = \frac{a}{e} \text{ and } x = -\frac{a}{e}$$

(3) Major axis: The line segment $A'A$ in which the foci F' and F lie is of length $2a$ and is called the major axis $(a > b)$ of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix $\left(\pm\frac{a}{e}, 0\right)$.

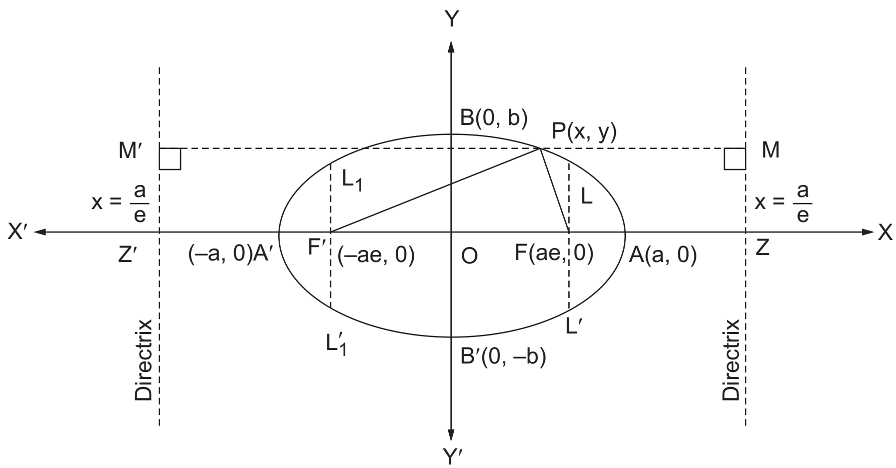


Fig. 3.19

(4) Minor axis: The y-axis intersects the ellipse in the points $B' \equiv (0, -b)$ and $B \equiv (0, b)$. The line segment $B'B$ of length $2b(b < a)$ is called the minor axis of the ellipse.

(5) Principal axes: The major and minor axis together are called principal axes of the ellipse.

(6) Centre: The point which bisects every chord of the conic drawn through it is called the centre of the conic $O \equiv (0, 0)$.The origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

(7) Latus rectum: The focal chord perpendicular to the major axis is called the latus rectum.

(*i*) Length of latus rectum $(LL') = \frac{2b^2}{a}$ (*ii*) Equation of latus rectum: $x = \pm ae$

(8) Focal radii: $SP = a - ex$ and $S'P = a + ex \Rightarrow SP + S'P = 2a$ = major axis.

(9) Eccentricity: $e = \sqrt{1-\frac{b^2}{a^2}}$

Remark: The sum of the focal lengths of any point on the ellipse is always equal to the major axis and is equal to $2a$. For all the above detailed terms, we refer Fig. 3.19.

Note:

(*i*) If distance of focus from the centre of ellipse = C then, $C = \sqrt{a^2-b^2}$

(*ii*) If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and nothing is mentioned, then it is assumed that $a > b$.

(*iii*) Ellipse is symmetric with respect to both the coordinate axes.

(*iv*) Foci always lie on major axis.

Example 13. Derive the equation of an ellipse whose focus is the point (–2, 2), whose directrix is the line $2x - 3y + 5 = 0$ and whose eccentricity is 1/3.

Solution: Let $P(x, y)$ be any point on the ellipse. Its focus is $S(-2, 2)$ and from P let PM be the perpendicular drawn to its directrix $2x - 3y + 5 = 0$. Then $SP = ePM$ or $SP^2 = e^2PM^2$

$$(x + 2)^2 + (y - 2)^2 = e^2\left[\frac{2x-3y+5}{\sqrt{(2)^2+(-3)^2}}\right]^2$$

$$\Rightarrow \quad (x + 2)^2 + (y - 2)^2 = \left(\frac{1}{3}\right)^2\left[\frac{2x-3y+5}{\sqrt{13}}\right]^2 = \frac{1}{(9\times 13)}(2x-3y+5)^2$$

$$\Rightarrow \quad (x + 2)^2 + (y - 2)^2 = \frac{1}{117}(2x-3y+5)^2$$

$$\Rightarrow \quad 117[x^2 + 4x + 4 + y^2 - 4y + 4] = [4x^2 + 9y^2 -12xy + 20x - 30y + 25]$$

$$\Rightarrow \quad 113x^2 + 113y^2 + 12xy + 448x - 438y + 911 = 0$$

which is the required equation of an ellipse

Example 14. If latus rectum of an ellipse is 1/2 of its minor axis, then which of the following is correct value of eccentricity '*e*'

(*i*) $\frac{5}{2}$ (*ii*) $\frac{2}{5}$ (*iii*) $\frac{\sqrt{3}}{2}$ (*iv*) $\frac{2}{\sqrt{3}}$

Solution: We are given that $\dfrac{2b^2}{a} = \dfrac{2b}{2}$ $\quad$ ($\because$ $2b$ is the value of minor axis)

$$\text{or} \qquad \frac{2b^2}{a} = b$$

$$\Rightarrow \qquad \frac{2b^2}{b} = a$$

$$\Rightarrow \qquad 2b = a \qquad \ldots(i)$$

Now squaring both sides of (i), we get

$$4b^2 = a^2$$

$$\Rightarrow \qquad 4a^2(1 - e^2) = a^2$$

$$\Rightarrow \qquad 1 - e^2 = \frac{1}{4}$$

$$\Rightarrow \qquad e^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore \qquad e = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \qquad \textbf{Ans. } (iii)$$

Example 15. Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 16x$.

Solution: Comparing the given equation with $y^2 = 4ax$, we get $4a = 16$

$$\Rightarrow \qquad a = 4$$

Thus focus of the parabola is (4, 0) and the equation of the directrix of the parabola is $x = -4$.

$$\text{Length of latus rectum} = 4a = 4 \times 4 = 16.$$

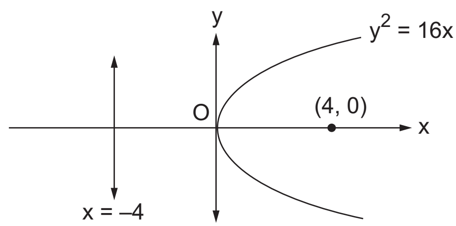


Fig. 3.20

Example 16. Find the equation of the parabola with focus (4, 0) and directrix $x = -4$.

Solution: The x-axis is the axis of parabola as focus (4, 0) lies on x-axis. Hence, the equation of parabola is of form $y^2 = 4ax$ or $y^2 = -4ax$. But it is given that directrix is $x = -4$ and focus (4, 0), the parabola must be of the form $y^2 = 4ax$ with $a = 4$

$\therefore$ required equation is $y^2 = 4 \times 4 \times x \Rightarrow y^2 = 16x$

Example 17. Find the coordinates of the foci, vertices, length of major axis, minor axis, eccentricity and latus rectum of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

Solution: Since $9 > 4$, $\therefore$ the major axis is along x-axis. Comparing given equation with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ we get

$$a = 3, \quad b = 2$$

$$\therefore \qquad \text{Eccentricity } e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{9 - 4} = \sqrt{5}$$

$\therefore$ Coordinates of foci are $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$; vertices are $(-3, 0)$ and $(3, 0)$; length of major axis $= 2 \times a = 6$ units

$$\text{Length of minor axis} = 2b = 4 \text{ units}$$

$$\text{Latus rectum} = \frac{2b^2}{a} = \frac{2 \times 4}{3} = \frac{8}{3}$$

Example 18. Find the equation of the ellipse whose vertices are $(\pm 10, 0)$ and foci are $(\pm 4, 0)$.

Solution: Since the vertices are on x-axis , the equation of ellipse will be of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where '$a$' is semi-major axis

It is given $a = 10$, $c = 4$

$\therefore$ Using $c^2 = a^2 - b^2 \Rightarrow b = \sqrt{100 - 16}$

$\Rightarrow \quad b = \sqrt{84}$

$\therefore$ Equation of ellipse is

$$\frac{x^2}{100} + \frac{y^2}{84} = 1$$

Example 19. Find the coordinates of the foci and the vertices, eccentricity, length of the latus rectum, of the hyperbola

$$\frac{x^2}{25} - \frac{y^2}{36} = 1$$

FUN FACT
The terms 'parabola' and hyperbola' were given by Greek mathematician Apollonius more than 2000 years ago!

Solution: Given

$$\frac{x^2}{25} - \frac{y^2}{36} = 1 \qquad \text{...(1)}$$

Comparing (1) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We get $a = 5$, $b = 6$ and $c = \sqrt{a^2 + b^2} = \sqrt{25 + 36} = \sqrt{61}$

$\therefore$ Coordinates of foci are $(\pm\sqrt{61}, 0)$ and that of vertices are $(\pm 5, 0)$

$$\text{Also eccentricity } e = \frac{c}{a} = \frac{\sqrt{61}}{5}$$

$$\text{Latus rectum} = \frac{2b^2}{a} = \frac{2 \times 36}{5} = \frac{72}{5}$$

Example 20. Find equation of hyperbola with foci $(0, \pm 2)$ and vertices $(0, \pm 1)$.

Solution: Since the foci is on y-axis, the equation of hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

As vertices are $(0, \pm 1)$, $a = 1$

Also as foci are $(0, \pm 2)$, $c = 2$

$\therefore \quad b^2 = c^2 - a^2 = 4 - 1 = 3$

$b^2 = 3$

$\therefore$ Equation of hyperbola is $\frac{y^2}{1} - \frac{x^2}{9} = 1$ is $9y^2 - x^2 = 9$

Applications of Coordinate Geometry

- Sketch of the building under construction uses pure coordinate geometry.
- Google maps use this concept in telling shortest possible path.
- Used in MS point for drawing curved and slanted lines.
- In astrology, for example knowing the positions of the planets (as their orbits are in same ecliptic plane) around the Sun.
- In determining the location of aircrafts accurately along with RADAR technology.
- Latitudes and longitudes are fully based on coordinate geometry.
- Military services.
- Map projections.
- Land measurements.

CASE STUDY

The coordinates of the boundary of a plot bought by farmer A satisfies the equation $x^2 + y^2 = 25$. Whereas the coordinates of plot of bought by farmer B satisfies the curve $\frac{x^2}{36} + \frac{y^2}{100} = 1$. Can you answer the following?

1. What is the shape of plot bought by farmer A?
2. Can you tell the parameters of the shape of plot bought by farmer A?
3. What is the shape of curve satisfied by the coordinates of boundary of plot bought by farmer B?
4. Can you give parameters of plot bought by farmer B?
5. Does the coordinate (0, 0) lies on the boundary of plot of farmer A?
6. Does the coordinate $\left(\frac{6}{\sqrt{2}}, \frac{10}{\sqrt{2}}\right)$ lies on the boundary of the plot of farmer B?

Check Out!!!!

Math Works

Download the MATLAB (Source: MathWorks) Free trial version after unit 5 is taught and check whether you can plot the following conics

1. $x^2 + y^2 = 1$
2. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
3. $y^2 = 4ax$
4. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

SUMMARY

1. **Straight line:**
   - Distance formula = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
   - Section formula = $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right) = \left(\frac{x_1 + kx_2}{k+1}, \frac{y_1 + ky_2}{k+1}\right)$
   - Area of $\Delta = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$

- Slope when 2 points (x_1, y_1) and (x_2, y_2), are given is $\tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$
- If lines are
  (A) Parallel then $m_1 = m_2$
  (B) Perpendicular then $m_1 m_2 = -1$
- Angle between two lines is $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

  If $\dfrac{m_2 - m_1}{1 + m_1 m_2}$ is positive then θ is acute
  If it is negative then θ is obtuse.
- Collinearity of points (A, B, C) means
  Slope of AB = Slope of BC

Forms of Equation of Line

- Point slope form (a point's coordinates and slope is given)
  $$y - y_0 = m(x - x_0)$$
- Two point form (coordinates of two points are given)
  $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
- Slope intercepts form (slope and intercept given)
  (A) y intercept c, $y = mx + c$
  (B) x intercept d, $y = m(x - d)$
- Intercept form - x intercept 'a' and y intercept 'b' are given
  then, $$\frac{x}{a} + \frac{y}{b} = 1$$
- Normal form (length of perpendicular from origin to line (P) and angle θ which normal makes with +ve direction of x-axis is given).
  $$x\cos\omega + y\sin\omega = P$$

General Equation of Line

is $$Ax + By + C = 0$$

- Slope-intercept form
  (A) $B \neq O$, $m = \dfrac{-A}{B}$ y-intercept $= \dfrac{-C}{B}$
  (B) $B = O$, m = undefined x-intercept $= \dfrac{-C}{A}$
- Intercept form
  (I) $C \neq O$, x-intercept $= \dfrac{-C}{A}$ y-intercept $= \dfrac{-C}{B}$
  (II) $C = O$, zero intercept on axes (passes through origin)

- Normal form $\dfrac{A}{\cos\theta} = \dfrac{B}{\sin\theta} = \dfrac{-C}{P}$

$$\cos\theta = \pm\frac{A}{\sqrt{A^2+B^2}},\ \sin\theta = \pm\frac{B}{\sqrt{A^2+B^2}},\ P = \pm\frac{C}{\sqrt{A^2+B^2}}\quad (P = +\text{ve})$$

2. **Circle:** Equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ where x, y are variables, g, f and c are constants and centre is $(-g, -f)$ *i.e.*,

$\left(-\dfrac{\text{coefficient of } x}{2}, \dfrac{-\text{ coefficient of } y}{2}\right)$ and radius $r = \sqrt{g^2 + f^2 - c}$.

- If (l, m) is the centre and r is the radius of the circle then its equation is given as $(x - l)^2 + (y - m)^2 = r^2$.
- Equation of a circle through three given points: $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$
- Equation of circle in diameter form: $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

3. **Parabola:** Standard equation of a parabola is

$$\boxed{y^2 = 4ax} \qquad \ldots(i)$$

where x and y are variables and a is constant.

From the equation (i) of parabola, we extract the following:

(i) Vertex is (0, 0)

(ii) Focus is $(a, 0)$

(iii) Axis is $y = 0$

(iv) Directrix is $x + a = 0$

4. **Hyperbola:** $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ Standard equation of hyperbola

where x and y are variables, a and b are constants and $b^2 = a^2(e^2 - 1)$.

If c is length of foci from centre then $c^2 = a^2 + b^2$ for hyperbola and $e = \dfrac{c}{a}$

5. **Ellipse:** Standard equation of ellipse referred to its principal axes along the co-ordinate axes is $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$ $= 1, a > b$ and $b^2 = a^2(1 - e^2) \Rightarrow a^2 - b^2 = a^2e^2$

- *Foci* $F \equiv (ae, 0)$ and $F' \equiv (-ae, 0)$
- Vertices - $A' \equiv (-a, 0)$ and $A \equiv (a, 0)$
- Equation of directrices - $x = \dfrac{a}{e}$ and $x = -\dfrac{a}{e}$
- If distance of focus from the centre of ellipse = C then, $C = \sqrt{a^2 - b^2}$

Exercises

Subjective Questions

Q.1. Evaluate the equation of the circle which touches both the axes in first quadrant with radius 1
[**Ans.** $x^2 + y^2 - 2x - 2y + 1 = 0$]

Q.2. If the slopes of two straight lines are 1/2 and 2, then find the angle of intersection between the two lines. $\left[\textbf{Ans. } \tan^{-1}\left(\frac{3}{4}\right)\right]$

Q.3. Let $P(5, 3)$ and $Q(10, 2)$ be two points. Find the slope of a line perpendicular to PQ.
[**Ans.** Slope = 5]

Q.4. An ellipse $9x^2 + 25y^2 = 225$ is given. Find the foci and eccentricity. $\left[\textbf{Ans. } \text{foci}(+4, 0); e = \frac{4}{5}\right]$

Q.5. Find the equation of hyperbola with eccentricity 3/2 and foci at $(\pm 1, 0)$. $\left[\textbf{Ans. } \frac{9x^2}{4} - \frac{9y^2}{5} = 1\right]$

Q.6. Find the equation of parabola with directrix at $x = -6$ and focus at $(6, 0)$. [**Ans.** $y^2 = 24x$]

Objective Questions

Q.1. The slope of the line passing through the points (5, 9) and (–2, 5) is
(*a*) 5 (*b*) 9 (*c*) 4/7 (*d*) 0 [**Ans.** (*c*)]

Q.2. The slope of line making an inclination of 45° with the positive direction of x-axis is
(*a*) $\sqrt{3}$ (*b*) 1 (*c*) $\frac{\sqrt{3}}{2}$ (*d*) ∞ [**Ans.** (*b*)]

Q.3. If A, B, C are three points in xy-plane then they are collinear iff
(*a*) Slope of OA = slope of BC (*b*) Slope of AB = slope of BC
(*c*) Slope of OC = slope of AB (*d*) Slope of OB = slope of AC [**Ans.** (*b*)]

Q.4. The equation of the lines parallel to the axes and passing through (5, 8) are
(*a*) x = 2, y = 8 (*b*) $x = 5, y = 3$ (*c*) $x = 5, y = 9$ (*d*) $x = 5, y = 8$
[**Ans.** (*d*)]

Q.5. The equation of line through (0, 1) with slope 2 is
(*a*) $y = 3x + 1$ (*b*) $2y = 2x + 1$ (*c*) $y = 2x + 1$ (*d*) $x = 3y + 1$
[**Ans.** (*c*)]

Q.6. Equation of circle with centre (0, 0) and radius 5 is
(*a*) $x^2 + y^2 = 9$ (*b*) $(x - 3)^2 + y^2 = 10$ (*c*) $x^2 + y^2 = 25$ (*d*) $x^2 + (y - 4)^2 = 9$
[**Ans.** (*c*)]

Q.7. An ellipse is the set of all points in a plane the sum of whose distances from two fixed points in the plane is
(*a*) Constant (*b*) Varies (*c*) 0 (*d*) 1 unit [**Ans.** (*a*)]

Q.8. Length of latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is

(*a*) $\frac{a^2}{2}$ (*b*) $\frac{b^2}{a}$ (*c*) $\frac{2b^2}{a}$ (*d*) 0 [**Ans.** (*c*)]

Mini Project

Form groups each comprising of five students .Try cutting off conic sections from ten right circular cones made from newspapers. Make a table for types/number of conic section each group makes. Compare your results and show it to your teacher.

Activity

Sketch your dream house on A3 size sheet using cutouts of all types of conic sections in a coordinate system. Make it attractive by coloring. Give a verbal presentation with your dreamhouse sheet (online/ offline) to your teacher explaining a few properties of conics too!.

Know More

Polar coordinate system

Let us consider (x, y) as cartesian coordinates of a point P in Fig. 3.21

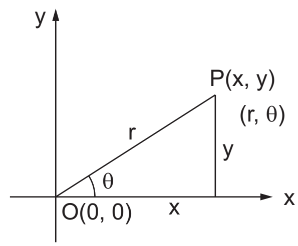


Fig. 3.21

Now write
$$x = r\cos\theta \quad ...(1)$$
$$y = r\sin\theta \quad ...(2)$$

Squaring both the sides of relations (1) and (2), we have,
$$x^2 = r^2\cos^2\theta$$
$$y^2 = r^2\sin^2\theta$$
$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2 \qquad (\because \cos^2\theta + \sin^2\theta = 1)$$
$$\therefore \quad r = \sqrt{x^2+y^2},\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x},\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

References and Suggested Readings

- ***Loney S.L.(1895),*** The Elements of Coordinate Geometry part I, Cartesian coordinates, Cambridge University press
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- Vasishtha A.R. & Agrawal D.C. (2019) Analytical Geometry 2D, Krishna's educational Publishers.
- ***Ballabh Ram, Saran Raghunath (13th edition)*** Co-ordinate Geometry, Prakashan Kendra, Lucknow
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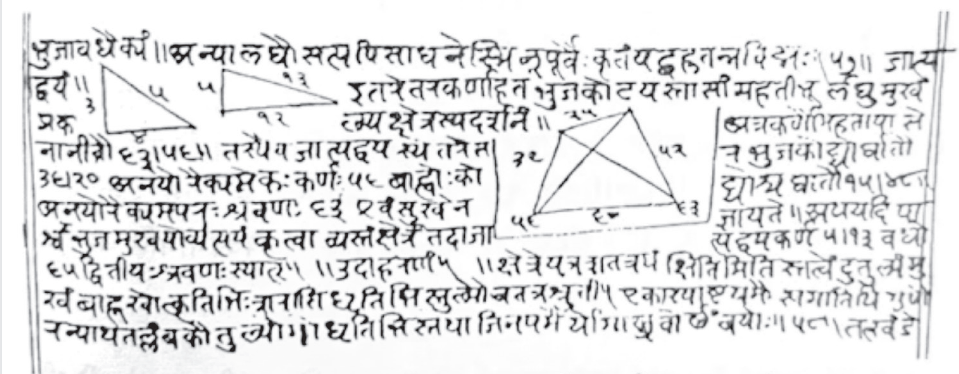


लीलावती के 'क्षेत्रव्यवहार' का एक नियम और उदाहरण ।

An excerpt from Bhaskaracharya's book 'Leelavati'
(**Source:** Muley Gunakar (1992), Sansaar ke Mahan Ganitagya, Raajkamal Prakashan)

4 Vector Algebra

UNIT SPECIFICS

This unit assimilates the concepts of rectangular resolution of a vector, addition and subtraction of vectors, scalar and vector products of 2 vectors and simple problems related to work, moment and angular velocity.

RATIONALE

Vector algebra comprises of study of mathematical objects called vectors (which have magnitude and direction both) under certain operations. It has evolved over a number of years to simplify matters. Vector algebra is comparatively easier to apply than geometry and requires knowledge of fewer rules. Many of the rules which we apply in basic algebra are applied in vector algebra too.

It is used widely in the field of mathematics, physics, engineering and many other areas. Displacement, velocity, acceleration, force, weight, electric field intensity, momentum etc. are a few examples of vectors. Vectors are widely used in force, torque, velocity, projectile, military, gaming, cricket etc. Hence, vectors have wide applicability and are much easier to handle than many other complex ways which make them very useful!

Pre-requisites

- Knowledge of basic algebra.
- Basics of work, moment and angular velocity.

Unit 4 Vector Algebra	Unit Outcomes(UO) By the end of the unit students will learn to -
U4-O1	Explain the concept of vectors, its rectangular resolution.
U4-O2	Do basic operations on vectors like-addition, subtraction, scalar multiplication.
U4-O3	Find out scalar product and cross product of two vectors.
U4-O4	Solve simple problems related to work, moment, angular velocity with reference to vectors.

CO-UO-Mapping

Unit 4 Outcomes	Expected mapping with course outcomes (1 - Weak correlation; 2 - Medium correlation; 3 - Strong correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U4-O1			1	3	1
U4-O2			1	3	1
U4-O3	1		1	3	1
U4-O4	1			3	1

4.1 INTRODUCTION

Every day we come cross many physical quantities like while measuring height of building its measurement could be 10 meters, which involves only magnitude (real number). Such quantities are called scalars. Whereas if we consider that how a hockey player should give a pass to another player of his team, then it involves magnitude (strength) and direction (another player's position) both. Such quantities are called vectors. In this unit we shall study briefly about vectors. Vectors can be usefully employed in handling various types of problem in mathematics, physics, technical branches etc. They facilitate study of quantities which possess direction too apart from magnitude. Physical quantities are broadly divided as follows:

1. **Vector quantities:** They have both magnitude and direction. For examples displacement, velocity, weight, force, angular velocity,moment etc.
2. **Scalar quantities:** They have only magnitude. For example – Mass, volume, work temperature etc.

Example 1. Classify the following as scalars and vectors

(*a*) 10 m North West (*b*) 10^{-10} Coulomb (*c*) 20 km/hr (*d*) 15 m/s towards East

(*e*) 100 Newton

Solution: (*a*) It is distance with direction, so is a vector.

(*b*) It is an electric charge, so is a scalar.

(*c*) It is speed, therefore a scalar.

(*d*) It is velocity of an object, so is a vector.

(*e*) It is a force, so is a vector.

FUN FACT

Both the terms 'vector' and 'scalar' are derived from Latin. Vector from *where* meaning 'to carry'; scalar from *scalaria* meaning 'ladder'.

Representation of Vectors

Recall: Any given part of a straight line with two end points - Initial (A) and terminal (B) is called a directed line segment (denoted as $\overrightarrow{AB}$)

A directed line segment is called a vector.

Every vector (Fig. 4.1) has the following three characteristics

1. **Length:** The length of $\overrightarrow{AB}$ will be denoted by $|\overrightarrow{AB}|$
2. **Support:** The line of unlimited length of which vector $\overrightarrow{AB}$ is a part is called its support.
3. **Sense:** It is from its initial point to the terminal point. That is, the sense of AB is from A to B and that of BA is from B to A.

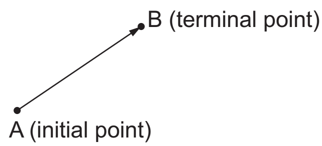


Fig. 4.1

Note: The vectors are denoted by putting an arrow over the symbols representing them like $\overrightarrow{AB}$, $\overrightarrow{OP}$ etc. Sometimes they are also represented by a single letter $\vec{a}$, $\vec{F}$ etc.

Example 2. Represent graphically

(*i*) A displacement of 80 km, 50° west of North

(*ii*) A displacement of 30 km North-East

Solution: Consider the Fig. (4.2)

(*i*) The vector $\overrightarrow{OP}$ represents required vector

(*ii*) The vector $\overrightarrow{OQ}$ represents required vector.

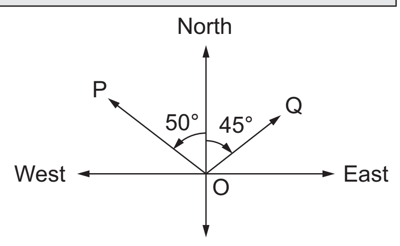


Fig. 4.2

Note: The magnitude *i.e.*, Distance between initial and terminal points of a vector is always a non-negative real number and is denoted as $|\overrightarrow{OA}|$ or $|\vec{a}|$ or $\boldsymbol{a}$. If $\vec{a} = x\hat{i} + y\hat{j}$ then $|\vec{a}| = \sqrt{x^2 + y^2}$.

4.2 Rectangular Resolution of a Vector

By resolution of a vector, it means determining the effect of a vector in a particular direction and the split vectors so obtained are known as components of the vector. If these components of a given vector are perpendicular (at 90°) to each other, they are called rectangular components. Let us consider an example of finding rectangular resolution of a vector $\vec{a}$ represented by $\overrightarrow{OA}$ as shown in Fig. 4.3. From the point O, two mutually perpendicular axis X and Y are drawn from the point A, two perpendicular AR and AS are dropped on X and Y-axis respectively. Then consider the right angled triangle ORA. Here the vector $\vec{a}$ makes an angle θ with position direction of X-axis. Therefore, we have $\cos\theta = \dfrac{OR}{OA}$.

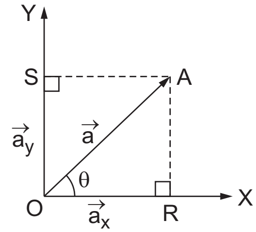


Fig. 4.3

$$\Rightarrow \qquad OR = OA\cos\theta$$

$$\Rightarrow \qquad \boxed{\vec{a}_x = \vec{a}\cos\theta} \qquad \text{...(1)}$$

$$\text{Similarly,} \qquad \boxed{\vec{a}_y = \vec{a}\sin\theta} \qquad \text{...(2)}$$

Thus we have resolved the vector $\vec{a}$ into two rectangular components $\vec{a}_x$ and $\vec{a}_y$ along X and Y axes respectively $\vec{a}_x$ is called the X-component of $\vec{a}$ and $\vec{a}_y$ is called the y-component of $\vec{a}$

Now, from (1) and (2) we get,

$$\cos\theta = \frac{a_x}{a} \quad \text{and} \quad \sin\theta = \frac{a_y}{a}$$

Squaring and adding both we get,

$$\sin^2\theta + \cos^2\theta = \frac{a_x^2}{a^2} + \frac{a_y^2}{a^2} \qquad \text{...(3)}$$

$$\text{But} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$\therefore \quad (3) \Rightarrow \qquad \frac{a_x^2 + a_y^2}{a^2} = 1 \quad \Rightarrow \quad a^2 = a_x^2 + a_y^2$$

$$\Rightarrow \qquad a = \sqrt{a_x^2 + a_y^2}$$

This given magnitude of $\vec{a}$ in terms of magnitudes of the components.

Note: If $\hat{i}$ and $\hat{j}$ denote vectors of unit magnitude along OX and OY axes respectively then

$$\vec{a}_x = a\cos\theta\hat{i} \quad \text{and} \quad \vec{a}_y = a\sin\theta\hat{j}$$

So that $\vec{a} = a\cos\theta\hat{i} + a\sin\theta\hat{j}$

Note: Unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$. It has length 1.

4.3 ALGEBRA OF VECTORS

The vectors have direction together with the magnitude, so their algebra is different than that of real numbers.

(*a*) **Addition of two vectors:** Let $\vec{a}$ and $\vec{b}$ be two vectors in a plane represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$ respectively.

Then their addition can be performed in following two ways

(1) Triangle law of addition of vectors

Here $\vec{a}$ and $\vec{b}$ are two vectors to be added. Under this law we draw a figure (Fig. 4.4) in which the initial point of $\vec{b}$ coincides with the terminal point of $\vec{a}$. The vector joining the initial point of $\vec{a}$ with the terminal point of $\vec{b}$ is the vector sum of $\vec{a}$ and $\vec{b}$.

i.e., $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\Rightarrow \quad \vec{a} + \vec{b} = \overrightarrow{OB}$

This method of addition of two vectors is called triangle law of addition of vectors.

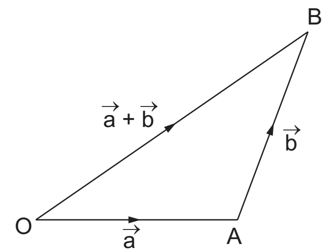


Fig. 4.4

> **Note**: Sum/resultant
>
> If $\vec{a} = a_1\hat{i} + a_2\hat{j}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j}$
>
> Then, $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j}$

(2) Parallelogram law of addition of vectors

Under this law, we draw the vectors $\vec{a}$ and $\vec{b}$ with both the initial points coinciding. Then we consider these two vectors as adjacent sides of a parallelogram. We complete the drawing of parallelogram. The diagonal so obtained from this parallelogram through the common initial point gives the sum of two vectors shown in Fig. 4.5

i.e., $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$

$\Rightarrow \quad \vec{a} + \vec{b} = \overrightarrow{OC}$

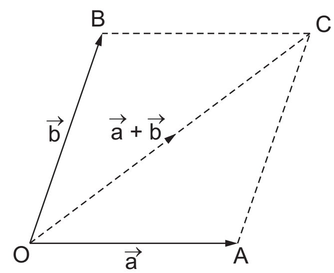


Fig. 4.5

This method of addition of two vectors is called parallelogram law of addition of vectors

Properties of Vector Addition

(1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (Commutative)

(2) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (associative)

(3) $\vec{a} + \vec{o} = \vec{a} = \vec{0} + \vec{a}$ (Additive identity)

(4) $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$ (additive inverse)

(*b*) **Multiplication of a vector by a scalar**: Let $\vec{a}$ be the given vector and *m* be a scalar. Then we define $\vec{b} = m\vec{a}$ as a vector of magnitude $|m\vec{a}|$. If m is positive, the direction of the vector $\vec{b} = m\vec{a}$ is same as that of $\vec{a}$ otherwise opposite to $\vec{a}$. This multiplication is called scalar multiplication. For example, if we multiply $\vec{a}$ by (–1) then its direction gets inverted. That is, $\vec{a}$ and $-\vec{a}$ have equal magnitudes but opposite directions.

> **Note:** If $\vec{a} = a_1\hat{i} + a_2\hat{j}$ then $m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j}$

(*c*) **Subtraction of vectors:**

Let $\vec{a}$ and $\vec{b}$ be two vectors. Then subtraction of these vectors $\vec{a}-\vec{b}$ is defined as sum of vectors $\vec{a}$ and $(-\vec{b})$. For this we invert the direction of $\vec{b}$ and add to $\vec{a}$ (Fig. 4.6)

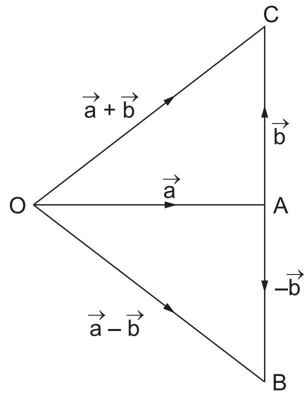


Fig. 4.6

> **Note**:
> 1. If $\vec{a} = a_1\hat{i}+a_2\hat{j}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}$ then, $\vec{a}-\vec{b} = (a_1-b_1)\hat{i}+(a_2-b_2)\hat{j}$
> 2. Two vectors are said to be equal if they have the same magnitude/components and the same direction, regardless of their initial points. For example: If $\vec{x} = a_1\hat{i}+b_1\hat{j}$ and $\vec{y} = a_2\hat{i}+b_2\hat{j}$ then $\vec{x}=\vec{y}$ iff $a_1 = a_2$ and $b_1 = b_2$.

Example 3. Evaluate a, b so that the vectors $\vec{x} = a\hat{i}+2\hat{j}$ and $\vec{y} = 4\hat{i}+b\hat{j}$ are equal.

Solution: Two vectors are equal if their corresponding components are equal.

Thus $\vec{x}$ and $\vec{y}$ will be equal if $a = 4$ and $b = 2$

Example 4. Let $\vec{a} = 2\hat{i}+4\hat{j}$, $\vec{b} = 4\hat{i}+2\hat{j}$. Is $|\vec{a}| = |\vec{b}|$? Are vectors $\vec{a}$ and $\vec{b}$ equal?

Solution: We have
$$|\vec{a}| = \sqrt{4+16} = \sqrt{20}$$
$$|\vec{b}| = \sqrt{16+4} = \sqrt{20}$$

So $|\vec{a}| = |\vec{b}|$. But the two vectors are not equal since their corresponding components are distinct.

Example 5. Find the unit vector in the direction of the sum of the vectors
$$\vec{a} = \hat{i}+2\hat{j} \quad \text{and} \quad \vec{b} = 4\hat{i}+5\hat{j}$$

Solution: The sum of given vectors $\vec{a}$ and $\vec{b}$ is $\vec{c} = \vec{a}+\vec{b} = 5\hat{i}+7\hat{j}$
$$|\vec{c}| = \sqrt{25+49} = \sqrt{74}$$

Thus the required unit vector is
$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{74}}[5\hat{i}+7\hat{j}]$$
$$\Rightarrow \qquad \hat{c} = \frac{5}{\sqrt{74}}\hat{i}+\frac{7}{\sqrt{74}}\hat{j}$$

> **FUN FACT**
> Vector analysis in its modern form was originally developed in the late 1800s for expressing the dynamics of physical quantities, like electric and magnetic fields

Example 6. Subtract the vector $\vec{b} = 4\hat{i}+9\hat{j}$ from $\vec{a} = 3\hat{i}+20\hat{j}$.

Solution: We have to find $\vec{a}-\vec{b}$
$$-\vec{b} = -4\hat{i}-9\hat{j}$$
$$\therefore \qquad \vec{a}-\vec{b} = (3-4)\hat{i}+(20-9)\hat{j} \quad \Rightarrow \quad \vec{a}-\vec{b} = -\hat{i}+11\hat{j}$$

Question 1. Write two different vectors having the same magnitude.

Question 2. Write two different vectors having the same direction.

> **Remark:** Any vector $\vec{a}$ in three dimensional systems can be expressed as $\vec{a} = x\hat{i}+y\hat{j}+z\hat{k}$, where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors parallel to x-axis, y-axis and z-axis here magnitude of $\vec{a}$ is $|\vec{a}| = \sqrt{x^2+y^2+z^2}$

4.4 TYPES OF VECTORS

1. **Zero/null vector:** A vector whose magnitude is zero is called a zero or null vector and is represented by 0.
2. **Co-initial vectors:** Two or more vectors having the same initial point are called co-initial vectors.
3. **Collinear vectors:** Two or more vectors are said to be collinear if they are parallel to the same line irrespective of their magnitudes and directions.
4. **Free vectors:** If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector.
5. **Coterminous vectors:** Vectors having the same terminal points are called coterminous vectors.

Note: Position vector of a point P (Fig. 4.7) is given as $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

where $\hat{i}$ - Unit vector along x-axis

$\hat{j}$ - Unit vector along y-axis

$\hat{k}$ - Unit vector along z-axis

$$\therefore \quad |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

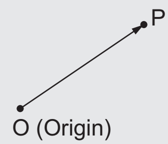


Fig. 4.7

Example 7. In the given Fig. 4.7, which vectors are:

(*i*) Collinear (*ii*) Collinear but not equal

(*iii*) Equal (*iv*) Co-initial

Solution: (*i*) $\vec{a}, \vec{b}, \vec{d}$ are collinear vectors.

(*ii*) $\vec{a}$ and $\vec{d}$ are collinear but not equal as their directions are not same.

(*iii*) $\vec{b}, \vec{d}$ are equal vectors.

(*iv*) $\vec{a}, \vec{b}, \vec{c}$ are co-initial vectors.

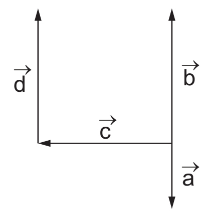


Fig. 4.8

4.5 PRODUCT OF TWO VECTORS

Product of two vectors is taken out by following two methods

1. Dot product or Scalar product:

The scalar product(or dot product) of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \cdot \vec{b} = ab \cos\theta \qquad ...(1)$$

where a and b are the magnitudes of $\vec{a}$ and $\vec{b}$ respectively and θ is the angle (Fig. 4.9) between them. It is scalar quantity. Scalar product is the magnitude of $\vec{a}$ multiplied by the projection of $\vec{b}$ onto $\vec{a}$(ab cos θ).The dot product between two vectors which are mutually perpendicular is zero (cos 90° = 0)

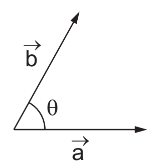


Fig. 4.9

Note:

1. The scalar product is commutative *i.e.*, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ and distributive *i.e.*, $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
2. If θ = 0, then $\vec{a} \cdot \vec{b} = ab$ and $\vec{a} \cdot \vec{a} = a^2$

3. If $\theta = \pi$, then $\vec{a} \cdot \vec{b} = -ab$
4. Angle θ between $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \vec{a} \cdot \vec{b}$ (from (1))

$$\Rightarrow \qquad \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right)$$

Scalar product in terms of components:

If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

then $\qquad \vec{a} \cdot \vec{b} = a_1a_2 + b_1b_2 + c_1c_2$

In particular, $\qquad \vec{a} \cdot \vec{a} = |\vec{a}|^2 = a_1^2 + b_1^2 + c_1^2$

Example 8. Evaluate 'θ' between the vectors

$$\vec{a} = \hat{i} + \hat{j} - 2\hat{k}, \quad \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Solution: The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{ab}$

As $\qquad \vec{a} \cdot \vec{b} = (1 \cdot 1) + (1 \cdot 2) + ((-2) \cdot 3) = 1 + 2 - 6 = -3$

and $\qquad a = |\vec{a}| = \sqrt{6}$

$\qquad b = |\vec{b}| = \sqrt{14}$

$$\therefore \qquad \cos\theta = \frac{-3}{\sqrt{2\times 3\times 7\times 2}}$$

$$\Rightarrow \qquad \theta = \cos^{-1}\left[\frac{-3}{2\sqrt{21}}\right]$$

Example 9. If $\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \lambda\hat{k}$, then find λ such that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal.

Solution: Consider $\qquad \vec{a} + \vec{b} = 2\hat{i} + (5+\lambda)\hat{k}$ and $\vec{a} - \vec{b} = -2\hat{j} + (5-\lambda)\hat{k}$

It is given that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal (90°)

$$\therefore \qquad (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$\Rightarrow \qquad (2 \cdot 0) + (0 \cdot (-2)) + (5 + \lambda)(5 - \lambda) = 0$$

$$\Rightarrow \qquad 25 - \lambda^2 = 0$$

$$\Rightarrow \qquad \lambda = \pm 5$$ **Ans.**

Applications of Dot Product

WORK DONE (MECHANICS).

A force acting on a particle is said to do work, if the particle is displaced in a direction which is not perpendicular to force. It is a scalar quantity.

Work done = Force × Displacement along the direction of force (Fig. 4.10).

Work done = $\vec{F} \cdot \vec{a} = Fa\cos\theta$.

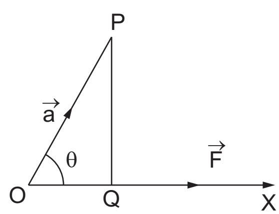


Fig. 4.10

Remark: Work done by a force is a scalar quantity. It is equal to the product of the magnitude of force and the resolved part of the displacement. If $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n$ are n forces acting on a particle then during the displacement d of the particle, work done by the n forces is $F_1 \cdot d, F_2 \cdot d, \ldots, F_n \cdot d$. That is these n forces will be replaced by its resultant force $\vec{R}$.

Example 10. Find the work done by the force $\vec{F} = \hat{i}+2\hat{j}+\hat{k}$ in the direction of displacement

$$\vec{a} = 3\hat{i}+4\hat{j}-\hat{k}$$

Solution: Given $\quad \vec{F} = \hat{i}+2\hat{j}+\hat{k}$ and $\vec{a} = 3\hat{i}+4\hat{j}-\hat{k}$

$\therefore$ Work done, $\quad W = \vec{F}\cdot\vec{a} \Rightarrow W = 3 + 8 - 1 \Rightarrow W = 10$ units.

Example 11. A force $\vec{F} = 2\hat{i}+5\hat{j}-\hat{k}$ acts at a point A. The point of application of force moves from the point A to point A', where $2\hat{i}+4\hat{j}+5\hat{k}$ and $3\hat{i}+5\hat{j}+\hat{k}$ are the position vectors of A and A' respectively. Find the work done.

Solution: Given $\quad \vec{F} = 2\hat{i}+5\hat{j}-\hat{k}$

Position vector of point $A = 2\hat{i}+4\hat{j}+5\hat{k}$

Position vector of point $A' = 3\hat{i}+5\hat{j}+\hat{k}$

$\therefore$ Displacement $\vec{a} = \overrightarrow{AA}$

$(3\hat{i}+5\hat{j}+\hat{k})-(2\hat{i}+4\hat{j}+5\hat{k}) = \hat{i}+\hat{j}-4\hat{k}$

Work done $W = \vec{F}\cdot\vec{a}$

$\Rightarrow \quad W = 2 + 5 + 4 \Rightarrow W = 11$ units

Example 12. Suppose a force of 10 N acts on an object in vertically upward direction and the object is displaced through 4m in vertically downward direction. Find the work done by the force during this displacement.

Solution: $\quad$ Work done $= \vec{F}\cdot\vec{a} = Fa\cos\theta$

Where θ is angle between the force $\vec{F}$ and the displacement $\vec{a}$

Here $\quad \theta = 180°$

Thus $\quad W = (10\text{ N})\,(4\text{m})\cdot\cos 180°$

$= -40$ N-m $= -40$ J **Ans.**

2. Cross product or vector product of two vectors

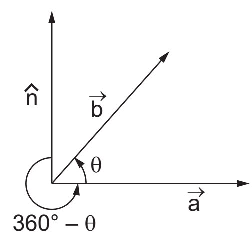


Fig. 4.11

The cross product or vector product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\times\vec{b}$, is itself a vector defined by $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$. The magnitude of this vector is $|\vec{a}\times\vec{b}| = ab\sin\theta$

Where a and b are the magnitudes of $\vec{a}$ and $\vec{b}$ respectively and θ is the smaller angle between the two vectors. $\hat{n}$ is unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ (Fig. 4.11). The direction of $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, in such a way that $\vec{a}$, $\vec{b}$ and this direction constitute a right-handed system (Fig. 4.12).

Let $\quad \vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

and $\quad \vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

Then, $$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

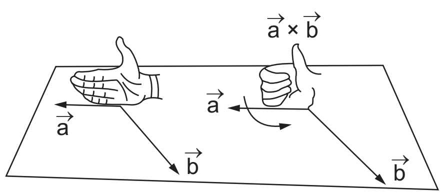


Fig. 4.12

Note:

1. $\vec{a} \times \vec{b} = 0$ iff $\theta = 0$ *i.e.*, $\vec{a}$ and $\vec{b}$ are parallel (or collinear) to each other.
2. If $\theta = \dfrac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\hat{n}$.
3. In terms of vector product, the angle between two vectors $\vec{a}$ and $\vec{b}$ is given by
$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$
4. Vector product is not commutative.
5. If $\vec{a}$ and $\vec{b}$ represent adjacent sides of a triangle then its area $= \dfrac{1}{2}|\vec{a} \times \vec{b}|$
6. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is $|\vec{a} \times \vec{b}|$.

Example 13. Find $|\vec{a} \times \vec{b}|$ if $\vec{a} = 2\hat{i}+\hat{j}+4\hat{k}$, $\vec{b} = 3\hat{i}+3\hat{j}+8\hat{k}$.

Solution:
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 4 \\ 3 & 3 & 8 \end{vmatrix}$$
$$= \hat{i}(8-12)-\hat{j}(16-12)+\hat{k}(6-3)$$
$$= -4\hat{i}-4\hat{j}+3\hat{k}$$
$$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{(-4)^2+(-4)^2+3^2} = \sqrt{16+16+9}$$
$$\vec{a} \times \vec{b} = \sqrt{41}$$

FUN FACT
The term *vector* was introduced by Irish mathematician Sir WR Hamilton.

Example 14. Find a unit vector perpendicular to each of the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ where $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$, $\vec{b} = 2\hat{i}+3\hat{j}+\hat{k}$.

Solution: We have
$$\vec{a} + \vec{b} = 3\hat{i}+4\hat{j}+3\hat{k}$$
$$\vec{a} - \vec{b} = -\hat{i}-2\hat{j}+\hat{k}$$

Vector $\vec{c}$ which is perpendicular to $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is $\vec{c} = (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 3 \\ -1 & -2 & 1 \end{vmatrix}$

$$\Rightarrow \quad \vec{c} = \hat{i}(4+6)-\hat{j}(3+3)+\hat{k}(-6+4)$$
$$\Rightarrow \quad \vec{c} = 10\hat{i}-6\hat{j}-2\hat{k}$$
$$\therefore \quad |\vec{c}| = \sqrt{100+36+4} = \sqrt{140} = 2\sqrt{35}$$

$\therefore$ Required unit vector is
$$\frac{\vec{c}}{|\vec{c}|} = \frac{5\hat{i}}{\sqrt{35}}-\frac{3\hat{j}}{\sqrt{35}}-\frac{\hat{k}}{\sqrt{35}} \quad \textbf{Ans.}$$

Question 3. Find area of a triangle having the points $A(1, 1, 2)$; $B(1, 3, 1)$ and $C(2, 2, 2)$ as its vertices.

Question 4. Find area of a parallelogram whose adjacent sides are given by the vectors $\vec{a} = 2\hat{i}+3\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+3\hat{k}$.

Remember:

1. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ $\qquad \hat{i}\times\hat{j} = \hat{k}$
   $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ $\qquad \hat{j}\times\hat{k} = \hat{i}$
   $\hat{j}\times\hat{i} = -\hat{k}$ $\qquad \hat{k}\times\hat{i} = \hat{j}$
   $\hat{k}\times\hat{j} = -\hat{i}$
   $\hat{i}\times\hat{k} = -\hat{j}$ $\qquad \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$
2. Parentheses are important for example: $(u \cdot v)w \neq u(v \cdot w)$

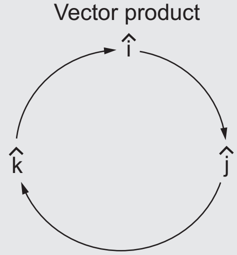


Fig. 4.13

Applications of Vector Product

Some important applications of vector product are:

1. **Moment of force ($\vec{M}$):** It is the rotational equivalent of linear force, also called torque ($\vec{\tau}$) or rotational force or turning effect. It is a force causing rotation around a specific point/axis, like a door rotating around its hinges. Let a force $\vec{F}$ be applied at a point P of a rigid body. Then moment of force $\vec{M}$ about a point, measures the tendency of force to turn the body about that point. If this tendency of rotation is in anti-clockwise direction,moment is positive, otherwise it is negative.

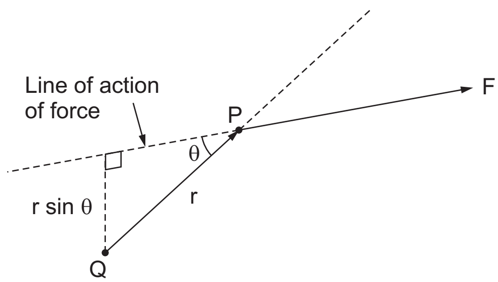


Fig. 4.14

Moment of a force $\vec{M}$ about a point O is $\vec{M} = \vec{r} \times \vec{F} = \vec{\tau}$ (Fig. 4.14)

Where $\vec{F}$ = force applied

P = point of application of force

Q = point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point of application of force w.r.t the point about which we want to determine the torque.

$$|\vec{\tau}| = rF\sin\theta$$

Where θ = angle between the direction of force and the position vector P w.r.t. Q.

$r\sin\theta$ = perpendicular distance of line of action of force from point Q, it is also called force arm.

$F\sin\theta$ = component of $\vec{F}$ perpendicular to $\vec{r}$.

The moment of a force about a point is a vector quantity and is always perpendicular to the plane of rotation of the body. S.I. unit is newton-metre (N.m).

Example 15. Find the moment about (1, 0, 1) of the force $2\hat{i}+3\hat{j}+5\hat{k}$ acting at (2, 1, –1).

Solution: Let $O \equiv (1, 0, 1)$

$A \equiv (2, 1, -1)$

and $\vec{F} = 2\hat{i}+3\hat{j}+5\hat{k}$ (Fig. 4.15)

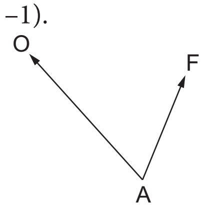


Fig. 4.15

Then, we know moment of force about O is given by $\overrightarrow{OA}\times\vec{F}$

Here $\overrightarrow{OA} = (2\hat{i}+\hat{j}-\hat{k})-(\hat{i}+\hat{k}) = \hat{i}+\hat{j}-2\hat{k}$

$$\therefore \qquad \overrightarrow{OA} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 2 & 3 & 5 \end{vmatrix} = \hat{i}(5+6) - \hat{j}(5+4) + \hat{k}(3-2) = 11\hat{i} - 9\hat{j} + \hat{k}$$

Example 16. Find the torque(moment of force) about point O and A due to given force $\vec{F}$.

Solution: Torque about point O,

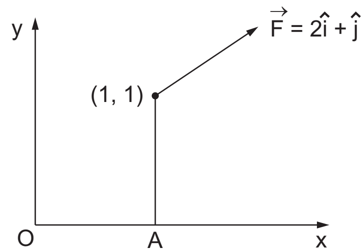


Fig. 4.16

$$\vec{\tau} = \vec{r}_0 \times \vec{F},\ \vec{r}_0 = \hat{i} + \hat{j},\ \vec{F} = 2\hat{i} + \hat{j}$$

$$\therefore \qquad \vec{\tau} = (\hat{i} + \hat{j}) \times (2\hat{i} + \hat{j}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} = \hat{k}(1-2) = -\hat{k}$$

Torque about point A, $\quad \vec{\tau} = \vec{r}_a \times \vec{F},\ \vec{r}_a = \hat{j}$ and $\vec{F} = 2\hat{i} + \hat{j}$

$$\Rightarrow \qquad \vec{\tau} = \hat{j} \times (2\hat{i} + \hat{j})$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} = -2\hat{k}$$

2. **Angular velocity:**

Angular velocity of an object in circular motion is defined as the rate of change of its angular displacement θ with respect to time.

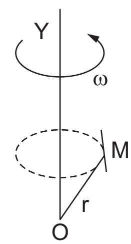


Fig. 4.17

In one dimension we talk about linear motion, linear displacement (x), linear velocity (v) where $v = \dfrac{dx}{dt}$. Similarly in two dimensions when we talk about circular motion with respect to unit time and θ as angular displacement then angular velocity, denoted by omega ω is, $\omega = \dfrac{d\theta}{dt}$. S.I. unit of angular velocity is radians per second.

Quite often the angular velocity is given in revolutions per second (rev/s). The conversion in radian per second may be made using 1 rev = 2π radian.

Relation between linear and angular velocity: When a rigid body rotates about a fixed line OY with an angular velocity ω, then the linear velocity v of the particle M is given by $v = \omega \times r$, where $r = \overrightarrow{OM}$ (position vector of the particle with respect to O) (Fig. 4.17).

$$\omega = |\omega| \times (\text{Unit vector along } OY)$$

Example 17. An object has an angular speed of 4 rad/s and the axis of rotation passes through the points (1, 2, 1) and (2, 2, –1). Find the velocity of the particle at point $M(2, 1, 3)$.

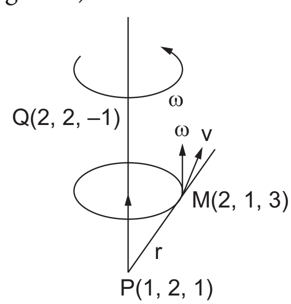


Fig. 4.18

Solution: Clearly
$$\overrightarrow{OP} = \hat{i} + 2\hat{j} + \hat{k}$$
$$\overrightarrow{OQ} = 2\hat{i} + 2\hat{j} - \hat{k}$$
$$\overrightarrow{PQ} = \hat{i} - 2\hat{k}$$
$$\Rightarrow \qquad |\overrightarrow{PQ}| = \sqrt{1+4} = \sqrt{5}$$

and $\vec{r} = \overrightarrow{PM} = (2\hat{i}+\hat{j}+3\hat{k})-(\hat{i}+2\hat{j}+\hat{k}) = \hat{i}-\hat{j}+2\hat{k}$

Now, $|\omega| = 4$ rad/s and unit vector along $\overrightarrow{PQ} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$ (Fig. 4.18)

$\therefore$ $$\omega = \frac{4}{\sqrt{5}}(\hat{i}-2\hat{k})$$

So, $$v = \omega \times \vec{r}$$

$$= \frac{4}{\sqrt{5}}(\hat{i}-2\hat{k})\times(\hat{i}-\hat{j}+2\hat{k})$$

$$= \frac{4}{\sqrt{5}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= \frac{4}{\sqrt{5}}[\hat{i}(-2)-\hat{j}(4)+\hat{k}(-1)]$$

$$v = \frac{4}{\sqrt{5}}(-2\hat{i}-4\hat{j}-\hat{k})$$

$$v = \frac{-4}{\sqrt{5}}(2\hat{i}+4\hat{j}+\hat{k}) \quad \textbf{Ans.}$$

Video Resource References [Source: NCERT]

Vectors and their representation

Vector Product

Scalar product of two vectors

Vector product-2

Application of Vector Algebra

- Dot and cross products are used in finding the work done and torque respectively.
- Volume of a parallelepiped can be calculated using vector algebra.
- It is used in the study of electromagnetism, hydrodynamics, blood flow, rocket launching, path of a satellite.
- Dot and cross products are used in calculating distance between two aircrafts in space and the angle between their paths.
- Vector dot product is used for calculations regarding installation of solar panels with reference to inclination of roofs and the direction of Sun, so as for maximum power generation.

CASE STUDY

A farmer has a field $ABCD$, were $A = (1, 1)$, $B = (3, 2)$, $C = (3, 3)$, $D = (2, 3)$

A child tries to fly a toy (paper plane) from the point $\vec{A}$, with velocity 100 cm/sec towards east. But breeze is blowing with a velocity of 40 cm/sec towards North. The toy flew 30 secs in $\overrightarrow{OP}$ direction with the resultant velocity. From $\vec{E}$ to $\vec{F}$ the toy flew for 10 secs with velocity of 100 cm/sec and finally dropped at point $\vec{F}$.

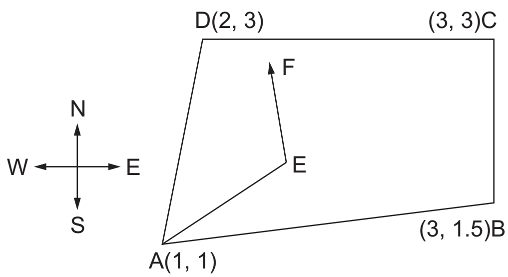


Fig. 4.19

Based on the above data, answer the following questions-

1. Find the position vectors of $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$.
2. What is the resultant velocity from $\vec{A}$ to $\vec{F}$?
3. Find the displacement from $\vec{A}$ to $\vec{F}$.
4. What is the resultant velocity and displacement from $\vec{E}$ to $\vec{F}$?

Check Out!!!!

Download the MATLAB (Source: MathWorks) free trial version after unit 5 is taught and check whether you can do the following operations on vectors $\vec{a} = 2\hat{i} + 3\hat{j}$ and $\vec{b} = \hat{i} + 4\hat{j}$

Math Works

(*i*) $\vec{a} + \vec{b}$ (*ii*) $\vec{a} - \vec{b}$ (*iii*) $\vec{a} \cdot \vec{b}$ (*iv*) $\vec{a} \times \vec{b}$

Check your results manually too!

SUMMARY

1. A directed line segment is called a vector.
2. If $\vec{a} = x\hat{i} + y\hat{j}$ then $|\vec{a}| = \sqrt{x^2 + y^2}$.
3. By resolution of a vector, it means determining the effect of a vector in a particular direction and the split vectors so obtained are known as components of the vector.

4. Triangle law of addition of vectors-

If $\vec{a} = a_1\hat{i}+a_2\hat{j}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}$.

Then, $\vec{a}+\vec{b} = (a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}$

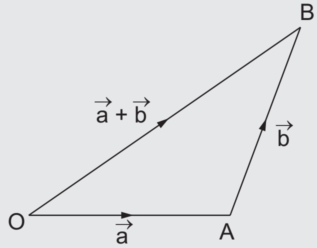


Fig. 4.20

6. **Properties of vector addition**

(1) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (Commutative)

(2) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associative)

(3) $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$ (Additive identity)

(4) $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$ (additive inverse)

7. **Multiplication of a vector by a scalar:** Let $\vec{a}$ be the given vector and m be a scalar. Then we define $\vec{b} = m\vec{a}$ as a vector of magnitude $|ma|$.

8. **Subtraction of vectors:** If $\vec{a} = a_1\hat{i}+a_2\hat{j}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}$, then, $\vec{a}-\vec{b} = (a_1-b_1)\hat{i}+(a_2-b_2)\hat{j}$

9. **Types of vectors:**

- **Zero/null vector:** A vector whose magnitude is zero is called a zero or null vector and is represented by 0.
- **Co-initial vectors:** Two or more vectors having the same initial point are called co-initial vectors.
- **Collinear vectors:** Two or more vectors are said to be collinear if they are parallel to the same line irrespective of their magnitudes and directions.
- **Free vectors:** If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector.
- **Coterminous vectors:** Vectors having the same terminal points are called coterminous vectors.

10. Position vector of a point is given as $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

11. Dot product or Scalar product- of two vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\cdot\vec{b} = ab\cos\theta$.

If $\vec{a} = a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\vec{b} = a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$, then $\vec{a}\cdot\vec{b} = a_1a_2+b_1b_2+c_1c_2$.

Application: Work done = $\vec{F}\cdot\vec{a} = Fa\cos\theta$.

12. Cross Product or Vector product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\times\vec{b}$, is itself a vector defined by $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$. The magnitude of this vector is $|\vec{a}\times\vec{b}| = ab\sin\theta$

Let, $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$. Then, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Applications:

- Moment of a force $\vec{F}$ about a point $O = \overrightarrow{OA}\times\vec{F}$, where A is any point on $\vec{F}$.
- Angular velocity $\nu = \omega \times r$

Exercises

Subjective Questions

Q.1. Position of a particle in a rectangular coordinate system is (3, 2, 5). Find out its position vector.
[**Ans.** $3\hat{i}+2\hat{j}+5\hat{k}$]

Q.2. If a particle moves from point $P(2, 3, 5)$ to point $Q(3, 4, 5)$. Find its displacement vector.
[**Ans.** $\hat{i}+\hat{j}$]

Q.3. Find the angle made by the vector $A = \hat{i}+\hat{j}$ with x-axis.
[**Ans.** 45°]

Q.4. An object of m kg with speed of v m/s, strikes a wall at an angle θ and rebounds at the same speed and same angle (Fig. 4.21). Calculate the magnitude of the change in momentum of the object. [**Ans.** $2mv\cos\theta$]

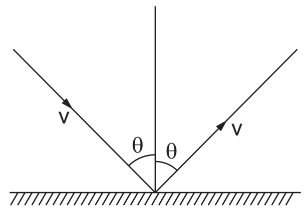


Fig. 4.21

Q.5. Prove that the three vectors
$\vec{A} = 3\hat{i}-2\hat{j}+\hat{k}$, $\vec{B} = \hat{i}-3\hat{j}+5\hat{k}$ and $\vec{C} = 2\hat{i}+\hat{j}-4\hat{k}$, form a right angled triangle.

Q.6. If $\vec{A}\times\vec{B} = |\vec{A}\cdot\vec{B}|$ then can you prove that angle between $\vec{A}$ and $\vec{B}$ is equal to 45°? Give reasons.

Objective Questions

Q.1. The torque of the force $\vec{F} = (2\hat{i}-3\hat{j}+4\hat{k})$ N acting at the point $\vec{r} = (3\hat{i}+2\hat{j}+3\hat{k})$ m is

(*a*) $(3\hat{i}-6\hat{j}-13\hat{k})$ N.m (*b*) $(17\hat{i}+6\hat{j}-\hat{k})$ N.m
(*c*) $(17\hat{i}-6\hat{j}-13\hat{k})$ N.m (*d*) 0 [**Ans.** (*c*) **Hint:** $\vec{\tau} = \vec{r}\times\vec{F}$]

Q.2. If $\vec{a} = 3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = 2\hat{i}-2\hat{j}+4\hat{k}$ then $\vec{a}\times\vec{b} = ?$

(*a*) 0 (*b*) $\hat{i}+\hat{j}+\hat{k}$ (*c*) $17\hat{i}+6\hat{j}+\hat{k}$ (*d*) $8\hat{i}-8\hat{j}-8\hat{k}$
[**Ans.** (*d*)]

Q.3. If angular velocity $\vec{\omega} = 3\hat{k}$ and radius $\hat{i} = 3\hat{j}$ then linear velocity is

(*a*) $\hat{i}$ (*b*) 0 (*c*) $-9\hat{i}$ (*d*) $7\hat{i}$ [**Ans.** (*c*)]

Q.4. If $\vec{F} = 2\hat{i}$, $\vec{r} = 3\hat{j}$, then moment of force is

(*a*) $3\hat{k}$ (*b*) 0 (*c*) $4\hat{j}$ (*d*) $-6\hat{k}$ [**Ans.** (*d*)]

Q.5. If the vectors $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b} = -4\hat{i}-6\hat{j}+\lambda\hat{k}$ are parallel to each other then value of λ is

(*a*) 0 (*b*) 4 (*c*) –9 (*d*) 2 [**Ans.** (*d*)]

Q.6. If the vector $2\hat{i}+3\hat{j}+8\hat{k}$ is perpendicular to the vector $4\hat{j}-4\hat{i}+\alpha\hat{k}$ then value of α is

(*a*) 1 (*b*) 1/3 (*c*) 1/2 (*d*) 3 [**Ans.** (*c*)]

Q.7. The magnitude of vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ is 3, 4 and 5 units respectively. If $\vec{A}+\vec{B} = \vec{C}$, then angle between $\vec{A}$ and $\vec{B}$ is

(*a*) 60° (*b*) 90° (*c*) 0° (*d*) 45° [**Ans.** (*b*)]

Q.8. If vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$ have magnitude 5, 12 and 13 units and $\vec{P}+\vec{Q}=\vec{R}$, then angle between $\vec{Q}$ and $\vec{R}$ is

(*a*) $\cos^{-1}\frac{3}{5}$ (*b*) $\cos^{-1}\left(\frac{12}{13}\right)$ (*c*) $\sin^{-1}\left(\frac{12}{13}\right)$ (*d*) $\sin^{-1}\left(\frac{5}{13}\right)$

[**Ans.** (*b*)]

Q.9. What vector must be added to the vectors $\hat{i}-2\hat{j}+2\hat{k}$ and $2\hat{i}+\hat{j}-\hat{k}$, so that the resultant will be a unit vector along *x*-axis?

(*a*) $(-2\hat{i}+\hat{j}-\hat{k})$ (*b*) $\hat{j}+\hat{k}$ (*c*) $3\hat{i}+\hat{k}$ (*d*) $2\hat{i}-\hat{j}+\hat{k}$

[**Ans.** (*a*)]

Q.10. If $\vec{A} = 4\hat{i}-3\hat{j}$ and $\vec{B} = 6\hat{i}+8\hat{j}$ then magnitude and direction of $\vec{A}+\vec{B}$ will be $5\sqrt{5}$ and $\tan^{-1}\left(\frac{1}{2}\right)$ respectively. TRUE/FALSE [**Ans.** True]

Mini Project

Make a group of five students each. Make a collage on vectors which must include the following:

(*a*) Pictorial definition of vectors with suitable examples.

(*b*) Detailed visual explanation of vector addition, subtraction, vector dot product and vector cross product.

(*c*) Visual presentation of rectangular resolution of a vector (2D).

Your subject teacher will evaluate and give a ***letter of appreciation*** to the group which performs the best!

Activity

Choose your favourite destination close to your home. Walk down to that place. Now create a scaled vector diagram of your stroll. The initial point of your first vector should be the starting point *i.e.*, your home. Connect the initial points of each successive vectors to the terminal point of the previous vector. Each directional distance vector to be made so as to fit on paper. Find out the resultant vector from starting point to the end point.

Know More

Polygon law of vector addition: It states that if number of vectors acting on a particle at a time are represented in magnitude and direction by various sides of a polygon taken in the same order, then their resultant is represented in magnitude and direction by the closing side of the polygon taken in the opposite order. This law is simply an extension of triangular law of addition. Geometrically, it is shown in Fig.

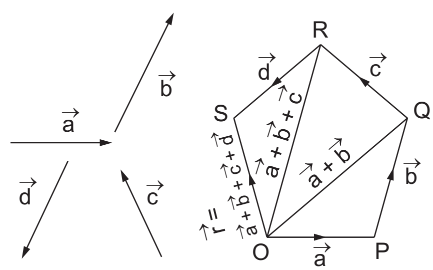


Fig. 4.22

$$\text{Resultant } \vec{r} = \vec{a}+\vec{b}+\vec{c}+\vec{d}\ (\overrightarrow{OS})$$

References and Suggested Readings

- Narayan Shanti, Mittal P.K. (1954) ,Vector Algebra , S. Chand & Co.
- NCERT (2007), Mathematics Text book XII Part II
- Chatterji P.N. (1998) ,Vector Algebra, Rajhans Press.
- Spiegel R Murray (1974), Vector Analysis, Schaum's Outline Series McGraw-Hill.

An excerpt of initial 12 verses "Brāhmasphutasiddhānta" authored by Brahmagupta. It has a total of 24 chapters and 1008 verses (on Mathematics and Astrology).

(**Source:** Muley Gunakar (1992), Sansaar ke Mahan Ganitagya, Raajkamal Prakashan)

5 Differential Equations

UNIT SPECIFICS

This unit elaborately discusses about solution of first order and first degree differential equation by variable separation method in a simple manner. An elementary introduction on MATLAB is given with appropriate pictorial examples.

RATIONALE

Differential equations are very important in mathematical modelling of various systems related to science & engineering, physics, chemistry, biology, economics and other disciplines. Systems which are dynamic *i.e.* change with respect to time (t) etc., are more often used in mathematical modelling. Also derivatives are nothing but synonymous to rate of change. And hence, the relations between derivatives and variables, which are called differential equations, are easier to model accordingly. Differential equations have a remarkable ability to predict the phenomena's around us.

MATLAB is a computing platform used by technical experts, engineers, scientists, and mathematicians all over the world. It has several features and functions, which we can avail of through its toolboxes. These toolboxes allows users to learn and apply specialized technology. MATLAB is a very developed platform and at the same time has very user friendly interface which makes it all the more applicable.

In this unit, we will study briefly about differential equations and MATLAB.

Pre-requisites

- Knowledge of calculus.
- Basics of computers.

Unit 5 Differential Equations	Unit Outcomes (UO) By the end of this unit students will learn to-
U5-O1	Conceptualize differential equations; find order and degree of differential equations.
U5-O2	Solve first order and first-degree differential equations with variable separable method.
U5-O3	Comprehend MATLAB; develop a perception about features of MATLAB.
U5-O4	Assimilate the basics of MATLAB; realize the advantages and disadvantages of MATLAB.

CO-UO Mapping

Unit 5 Outcomes	Expected mapping with course outcomes (1 - Weak correlation; 2 - Medium correlation; 3 - Strong correlation)				
	CO -1	CO-2	CO-3	CO-4	CO-5
U5-O1		1		3	

U5-O2		1		3	
U5-O3	1				3
U5-O4					3

5.1 DIFFERENTIAL EQUATION

You are familiar with the concept of an equation which is a mathematical statement with an "equal to" symbol between two expressions with equal values. For example: $x^2 + 2x + 1 = 0$; $4 \sin x + \tan x = 0$, $3x + 2y = 4$ etc. In this unit, we will study equations which involve derivatives as their terms apart from variables. Such equations are called differential equations. From differential calculus you learnt how to find out derivative of a function and from integral calculus, you know how to find a function whose derivative is given. So, combining these we would study some basic concepts related to differential equations.

5.2 BASIC DEFINITIONS/CONCEPTS

An equation which contains the derivatives of the dependent variables with respect to independent variables is known as a ***differential equation***. Depending upon the number of independent variables we can classify them as following:

(*i*) ***O.D.E. – Ordinary Differential Equations***

Such equations contain derivative of a dependent variable with respect to only one independent variable. For example: $\frac{dy}{dx} + 4y = \sin x$.

Here, dependent variable is y and independent variable is x.

(*ii*) ***P.D.E. – Partial Differential Equations***

Such equations contain derivative of a dependent variable with respect to two or more independent variables. For example: $\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y} = 10$.

Here z is the dependent variable and x, y are independent variables.

In this unit we confine our study to O.D.E. of first order and first degree.

Note: In general, following notations are used for derivatives:

$$\frac{dy}{dx} = y' \text{ or } y_1;\ \frac{d^2y}{dx^2} = y'' \text{ or } y_2;\ \frac{d^3y}{dx^3} = y''' \text{ or } y_3 \ldots \frac{d^ny}{dx^n} = y_n.$$

5.2.1 Order and Degree of a Differential Equation

Order of a differential equation is defined as the order of the highest derivative occurring in the equation.

Whereas, ***Degree*** of a differential equation is defined as the power of the highest order derivative occurring in the polynomial equation of derivatives after the equation is made free from radical signs and fractions.

FUN FACT
Differential equations are special because the solution of a differential equation is itself a function instead of a number.

For Example:

S. No.	Differential Equation	Order	Degree
1.	$\frac{dy}{dx}+y=e^x$	1	1
2.	$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\sin x$	2	1
3.	$\left(\frac{d^3y}{dx^3}\right)^2+\frac{d^2y}{dx^2}+y=10x$	3	2
4.	$\sqrt{\frac{d^2y}{dx^2}}=\frac{d^3y}{dx^3}$	3	2
5.	$\left(\frac{dy}{dx}\right)^2+\cos^2 y=0$	1	2

Remark: Order and degree of a differential equation are always positive integers.

Example 1. Find the order and degree (if defined) of the following differential equation

(1) $\frac{dy}{dx}+4y=\sin x$ (2) $\frac{d^2y}{dx^2}+xy\frac{dy}{dx}+2y=0$

(3) $\frac{d^3y}{dx^3}+\sin\left(\frac{d^2y}{dx^2}\right)+y=\cos x$ (4) $\sqrt{\frac{dy}{dx}}=\frac{d^3y}{dx^3}$.

Solution:

(1) $\frac{dy}{dx}$ is the highest order derivative, so order is 1. It is a polynomial equation in $\frac{dy}{dx}$ and highest power of $\frac{dy}{dx}$ is one, so degree is one.

(2) Here highest order derivative $=\frac{d^2y}{dx^2}$ $\therefore$ order = 2 and it is a polynomial equation of derivatives and power of $\frac{d^2y}{dx^2}$ is one, so degree = 1.

(3) Here, highest order derivative $=\frac{d^3y}{dx^3}$ $\therefore$ order = 3 but it is not a polynomial equation of derivatives, as $\sin\left(\frac{d^2y}{dx^2}\right)$ term is present. So degree is not defined.

(4) Making the given O.D.E. free from radical signs, it reduces to $\left(\frac{d^3y}{dx^3}\right)^2=\frac{dy}{dx}$.

Here, highest order derivative $=\frac{d^3y}{dx^3}$ $\therefore$ order = 3 and it is a polynomial equation of derivatives; also power of $\frac{d^3y}{dx^3}$ is two so degree = 2.

5.2.2 Solution of an Ordinary Differential Equation

Any function/curve is called the solution of a differential equation if it satisfies the equation. [*i.e.*, if left hand side is equal to right hand side of the equation.(L.H.S. = R.H.S.)].

The solution which consists of arbitrary constants is called the *general solution (or integral or primitive*) of the differential equation. Whereas a solution which is free from arbitrary constants is called a *particular solution* of the differential equation.

A differential equation may have a unique solution or many solutions or no solution. The general (or complete) solution of an n^{th} order differential equation will have n arbitrary constants.

Example 2. The number of arbitrary constants in the general solution of a differential equation of fifth order are:

(*a*) 0 (*b*) 3 (*c*) 4 (*d*) 5

Solution: Number of arbitrary constant in general solution = the order of the differential equation = 5.

Ans. (*d*)

Example 3. Find out the number of arbitrary constants in the particular solution of a differential equation of order 10.

Solution: Number of arbitrary constants in any particular solution of a differential equation = 0

$\therefore$ Answer is zero.

FUN FACT

The date of birth of differential equations is taken to be November, 11, 1675, when German mathematcian Gottfried Wlthelm Leibnitz first put in black and white the identity $\int y\,dy = \frac{1}{2}y^2$. He formulated *'method of separation of variables'*, '*method of solving the homogeneous differential equations of the first order*' and '*method of solving a linear differential equation of the first-order*' all within 25 years of the birth of differential equations—all by a single man!

Example 4. Verify that the function $y = A \sin x + B \cos x$ is a general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ (A, B are constants)

Solution: Given differential equation is $\frac{d^2y}{dx^2} + y = 0$ ...(1)

$\therefore$ $y = A \sin x + B \cos x$ will be a general solution of (1) if its substitution would give L.H.S. = R.H.S.

Consider, differentiating $y = A \sin x + B \cos x$ ...(2)

We get $\frac{dy}{dx} = A \cos x - B \sin x \Rightarrow \frac{d^2y}{dx^2} = -A \sin x - B \cos x$ ...(3)

Substituting the values of y and $\frac{d^2y}{dx^2}$ from (2) and (3) to (1) we get

$$\begin{aligned} \text{L.H.S.} &= -A \sin x - B \cos x + A \sin x + B \cos x \\ &= 0 = \text{R.H.S.} \quad \text{Hence verified.} \end{aligned}$$

Recall:

1. Function is a relation in which each element of the domain is associated with exactly one element of the co-domain.
2. Derivative is a measure of how a function changes as its input changes.

5.2.3 Formation of a Differential Equation Whose General Solution is Given

Let the general solution given be

$$f(x, y, a) = 0 \quad ...(1)$$

where a is an arbitrary constant, x is the independent variable and y is the dependent variable. To obtain corresponding differential equation, we have to eliminate a, for which we need two equations. One is given by (1) and the other we obtain by differentiating (1) w.r.t. 'x'. As a result we obtain $g\left(x, y, \frac{dy}{dx}\right) = 0$ which is the required differential equation (it represents a family of curves).

Similarly, we can extend the above method for two, three and more arbitrary constants to get the desired differential equation. (We need as many equations as the number of arbitrary constants) Here the order of the differential equation so obtained, representing a family of curves, is same as the number of arbitrary constants present in the equation corresponding to the family of curves.

Example 5. Form the differential equation representing the family of curves $y = ax$, where a is arbitrary constant.

Solution: Given equation is

$$y = ax \quad ...(1)$$

Differentiating (1) w.r.t. 'x' we get

$$\frac{dy}{dx} = a \quad ...(2)$$

$\therefore$ (1) and (2) together give

$$y = x\frac{dy}{dx} \quad \Rightarrow \quad x\frac{dy}{dx} - y = 0$$ which is the required differential equation.

5.3 SOLUTION OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATION BY VARIABLE SEPARATION METHOD

A first order and first degree differential equation can be written as

$$\frac{dy}{dx} = \phi(x, y) \quad ...(1)$$

There are many methods for solving O.D.E. of type (1). Here we will be studying in detail only one method *i.e.*, *variable separation method*.

If (1) can be written in the form

$$f(x)dx = g(y)dy \quad ...(2)$$

Then we say that the variables are separable. We solve such differential equations by integrating on both the sides *i.e.*, solution is

$$\int f(x)dx = \int g(y)dy + c$$

where c is the constant of integration.

Example 6. Solve $(1 + x^2)dy = (1 + y^2)dx$.

Solution: Given differential equation is $(1 + x^2)dy = (1 + y^2)dx$.

$$\Rightarrow \quad \frac{dy}{1+y^2} = \frac{dx}{1+x^2} \quad \text{(Separating the variables)}$$

Now integrating both sides, we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + c \quad \Rightarrow \quad \tan^{-1} y = \tan^{-1} x + c \quad \textbf{Ans.}$$

Example 7. Evaluate: $\dfrac{dy}{dx} = e^{x-2y} + x^4 e^{-2y}$.

Solution: The given differential equation is

$$\frac{dy}{dx} = e^{x-2y} + x^4 e^{-2y} = (e^x + x^4)e^{-2y}$$

Now, separating the variable, we get

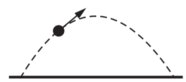
$$(e^x + x^4)dx = e^{2y}dy$$

Integrating both sides, we get

$$\int (e^x + x^4)dx = \int e^{2y} dy + c$$

$$\Rightarrow \quad e^x + \frac{x^5}{5} = \frac{e^{2y}}{2} + c \quad \textbf{Ans.}$$

FUN FACT



The path followed by a projectile launched from a cannon follows a curve determined by an ordinary differential equation that is derived from Newton's second law.

Example 8. $\log\left(\dfrac{dy}{dx}\right) = a_1 x + a_2 y$.

Solution: The given differential equation is $\log\left(\dfrac{dy}{dx}\right) = a_1 x + a_2 y$.

$$\Rightarrow \quad \frac{dy}{dx} = e^{a_1 x + a_2 y}$$

$$\Rightarrow \quad \frac{dy}{e^{a_2 y}} = e^{a_1 x} dx \quad \text{(Separating the variables)}$$

Integrating both sides, we get $\displaystyle\int \frac{dy}{e^{a_2 y}} = \int e^{a_1 x} dx + c$

$$\Rightarrow \quad \frac{e^{-a_2 y}}{-a_2} = \frac{e^{a_1 x}}{a_1} + c \quad \Rightarrow \quad \frac{e^{a_1 x}}{a_1} + \frac{e^{-a_2 y}}{a_2} + c = 0 \quad \textbf{Ans.}$$

Example 9. Solve $5xdy - 2ydx = 2x^2dy$.

Solution: The given differential equation is $5xdy - 2ydx = 2x^2dy$.

Separating the variables, we get

$$(5x - 2x^2)dy = 2ydx$$

$$\Rightarrow \quad \frac{dy}{2y} = \frac{dx}{x(5-2x)} \quad \Rightarrow \quad \frac{dy}{2y} = \frac{1}{5}\left[\frac{1}{x} + \frac{2}{5-2x}\right]dx$$

Integrating both sides, we get

$$\int \frac{dy}{2y} = \frac{1}{5}\int \left[\frac{1}{x} + \frac{2}{5-2x}\right]dx + \log c$$

$$\Rightarrow \quad \frac{1}{2}\log y = \frac{1}{5}\left[\log x + \frac{2}{-2}\log(5-2x)\right] + \log c$$

$$\Rightarrow \qquad \log\sqrt{y} = \frac{1}{5}\left[\log\left(\frac{x}{5-2x}\right)\right] + \log c$$

$$\Rightarrow \qquad \sqrt{y} = c\left(\frac{x}{5-2x}\right)^{1/5} \qquad \textbf{Ans.}$$

5.4 MATLAB – AN INTRODUCTION

The abbreviation MATLAB stands for MATrix LABoratory. MATLAB is a high level multi-paradigm language meant for technical and mathematical computing. It was created by ***Cleve Moler***, initially as a teaching tool for mathematics in the 1970's. Later in 1980's MATLAB was released as a commercial product. It has an interactive environment with hundreds of built-in-functions for technical computations, graphical multi-domain simulations, animations etc. MATLAB is used all over the world especially by engineers, scientists, technicians to analyze data, develop algorithms and create models.

Logo of MATLAB

[Source: www.mathworks.com]

The MATLAB logo is a three-dimensional L-shaped membrane. It is an eigenfunction of the wave equation. (The wave equation is a fundamental model in mathematical physics that describes how a disturbance travels through matter.)

5.4.1 Salient Features

Some of the salient features of MATLAB are-

- Basic building block of MATLAB is the matrix.
- **Live Editor:** It has live editor for creating scripts that combines code, output and formatted text in an executable notebook.

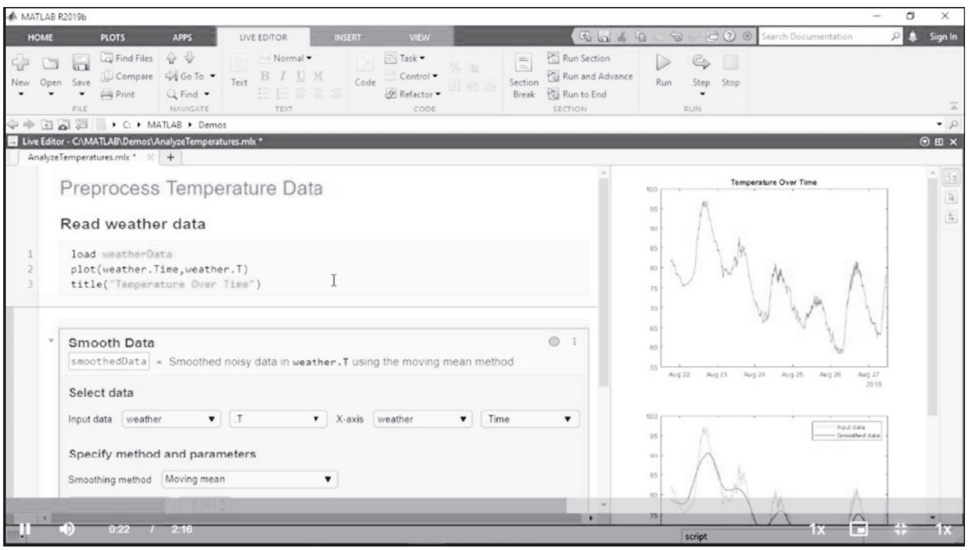


Fig. 5.1: (Live Editor)
(*source:www.mathworks.com*)

- Platform Independent: It is supported by windows, Macintosh, Linux, Unix etc.
- Program written on any of these platforms in MATLAB will run on the other too.

- Its toolboxes are professionally developed and fully documented.
- MATLAB has interactive apps through which we can iterate our data until we get desired results, thereafter generating MATLAB programs automatically related to our work.
- It scales one's analyses and there is no need to rewrite code or learn big data programming (or memorize complicated techniques.)
- Data Analysis: Thousands of prebuilt functions in MATLAB can be used to organize, clean and analyze complex date sets (which can be imported too!) in diverse fields like finance, medical etc. This can thereafter be documented using MATLAB Live Editor and exported in PDF, MS Word, Latex and HTML formats.
- Graphics: We can visualize the data by using built-in plots in MATLAB. This helps in identifying the underlying patterns and trends. These can be exported and shared too!

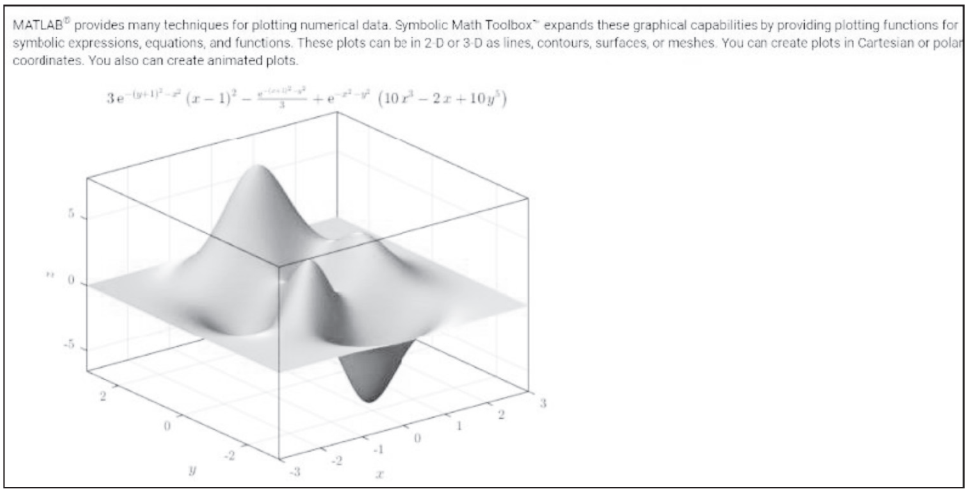


Fig. 5.2: (Graphics)
(*source:www.mathworks.com*)

- Algorithm development: It provides us the tools to transform our ideas into algorithms much faster than other languages like C, C++ or Fortran. These algorithms can be tested & verified, shared & distributed as well as deployed in larger systems.
- App Building: App designer in MATLAB allows one to create professional app (desktop & web apps) without being a professional software developer. These apps can be(royalty free) shared too!

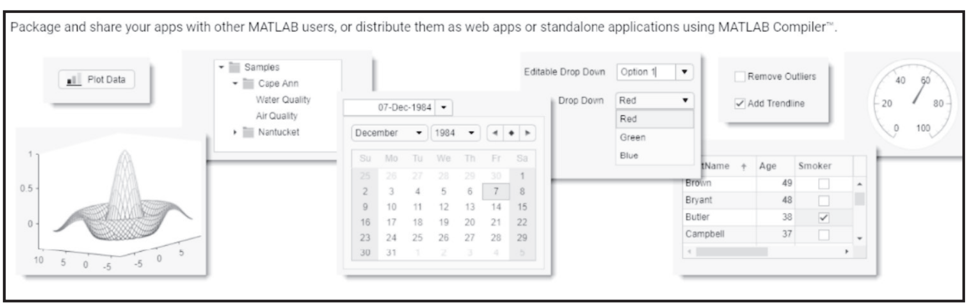


Fig. 5.3: (App Building)
(*source:www.mathworks.com*)

- MATLAB can be used with other languages too, like C,C++, Fortran, Java, Python, COM components and applications (.NET) etc. This feature helps in team work, as teams using different programming languages can work together.
- Parallel computing: This toolbox helps in performing large–scale computations using multicore desktops, graphic processing units (GPU's), computer clusters. Simulations which took months, run via this tool in a few days.

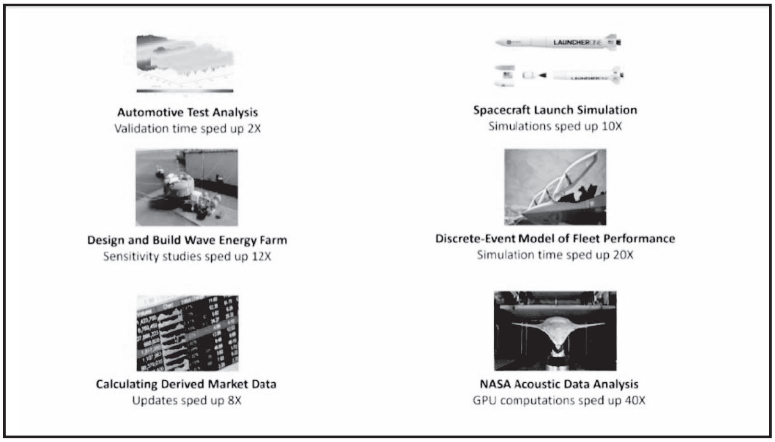


Fig. 5.4: (A few parallel computing toolboxes)
(*source:www.mathworks.com*)

- Web and desktop deployment: This application helps to share work one does in MATLAB with people who do not have access to MATLAB.

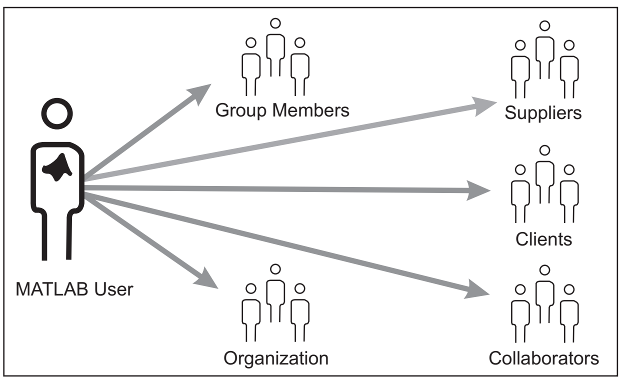


Fig. 5.5: (Use of application deployment to share MATLAB programs)
(*source:www.mathworks.com*)

- Use MATLAB in the cloud: One can use MATLAB in a web browser without installing, configuring or managing any software. MATLAB drive helps to store, access and work with one's files from anywhere.

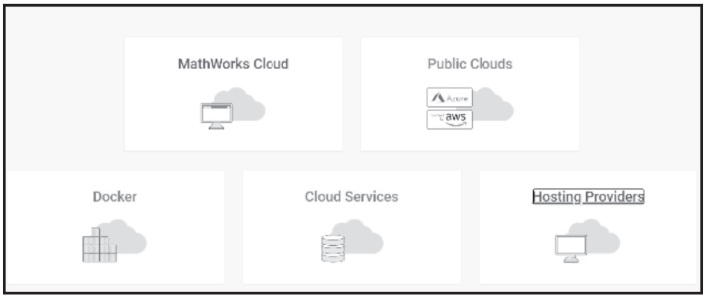
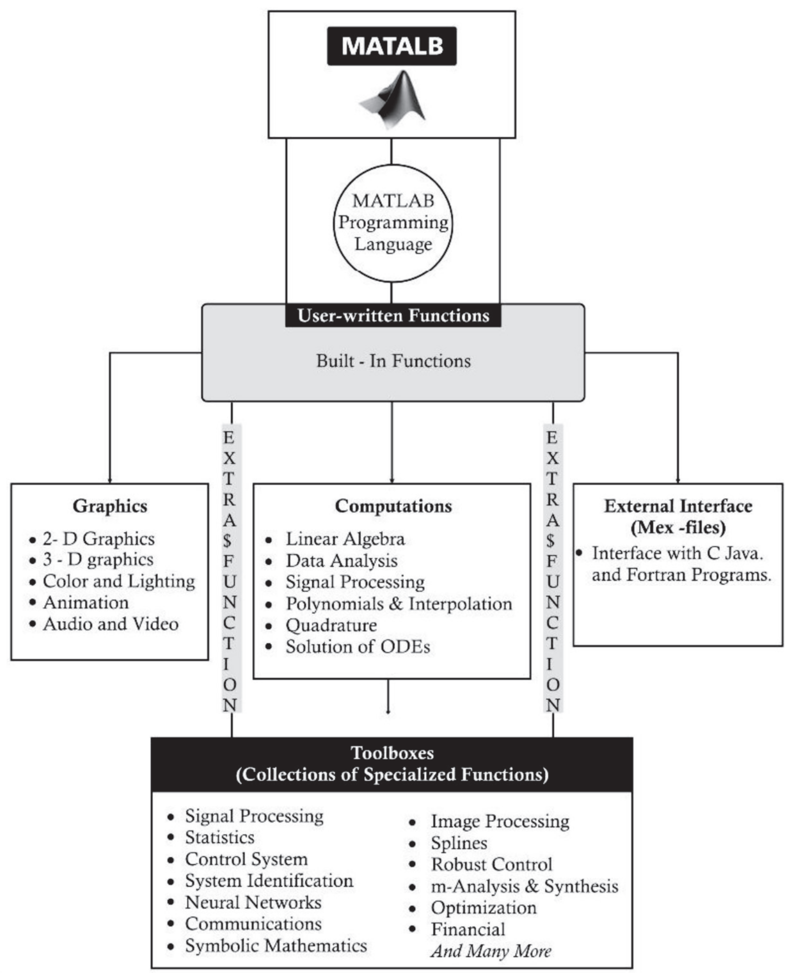


Fig. 5.6: (Runs in various cloud environments)
(*source:www.mathworks.com*)

- A figure of some features of MATLAB is give below:



A schematic diagram of MATLAB's main features.
Source: Getting started with Matlab by Rudra Pratap

Fig. 5.7

5.4.2 Basics OF MATLAB

On nearly all platforms, MATLAB works through following basics:-

1. **MATLAB desktop:** It is the main place where we work. It consists of the following sub windows:
   (*a*) **Command Window:** It is the main window and all commands are typed in this window at the MATLAB prompt(>>).
   (*b*) **Current Directory Pane:** Here all the files from current directory are listed.
   (*c*) **Details (File) Pane:** It is below the current directory pane and shows details of file selected in current directory pane.
   (*d*) **Workspace Pane:** It lists all variables generated by the user.
   (*e*) **Command History Pane:** All commands types in command window get recorded here.

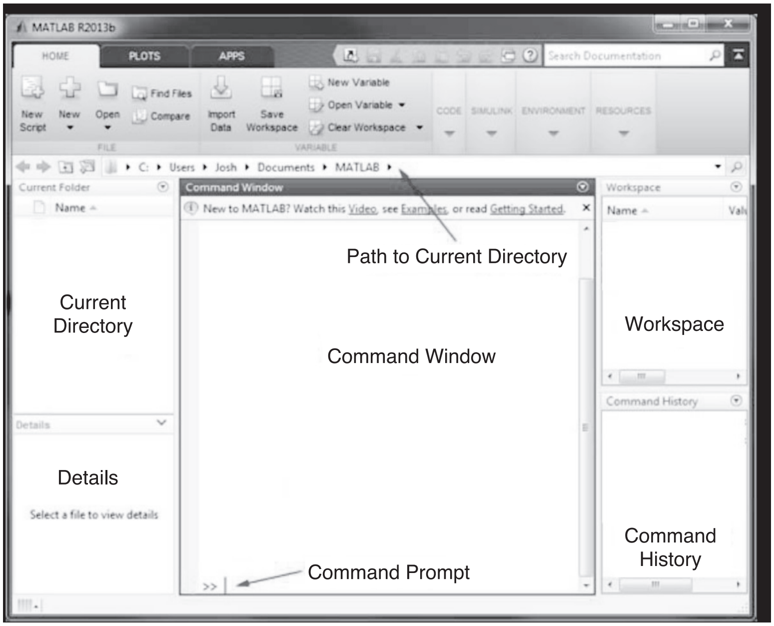


Fig. 5.8: (MATLAB Desktop)
(Source: www.mathworks.com)

2. **Figure Window:** The output of all graphics commands typed in command window are stored here.
3. **Editor Window:** Here the user can write, edit, create and save programs in files called M-files.
4. **ON-LINE HELP:** MATLAB has help option for all its function and also has demonstration programs too for explaining its features.
5. **Input-Output:** It supports interactive computation. The fundamental data type in MATLAB is an array/matrix. Mentioning dimensions of matrix is not needed. MATLAB is case-sensitive. The output of every command is shown on the screen unless it is directed otherwise.
6. **File Types:** MATLAB reads and writes several types of files. We study here following five types
   (*i*) **M-Files:** They have **.m** extension and are of two types: (*a*) Script files and (*b*) Function files. Most of the programs are saved as M-files.

(*ii*) **Mat-Files:** They have **.mat** extension. These files are created by MATLAB, when user saves data with the 'save' command.

(*iii*) **Fig-Files:** They have **.fig** extension. These are created by saving a figure in this format.

(*iv*) **P-Files:** They have **.p** extension. These are compiled M-files.

(*v*) **Mex-files:** They have **.mex** extension.

7. **Quitting MATLAB:** To end MATLAB session, type quit in the command window or select the file "Exit" MATLAB in the desktop main menu.
8. For elaborate details on MATLAB see the official website of Mathworks.

Remark: *Toolboxes of MATLAB:* These are a collection of numerous functions built on the MATLAB computing environment. Like - curve fitting toolbox, Create 2D plots, Fourier Transforms etc.

Example 10. Given an example of adding scalar to an array.

Solution: Create an array A and add scalar value 4 to it.

$$
\begin{aligned}
>> A &= [0, 2; 2, 0] \\
>> C &= A + 4 \\
C &= 4 \quad 6 \\
& 6 \quad 4
\end{aligned}
$$

The scalar value is added to each entry of A.

Example 11. Give an example of appending strings.

Solution: Create two 1 by 3 string arrays and then append similarly located strings in the arrays.

$$
\begin{aligned}
>>a1 &= [\text{"White" "Black" "Brown"}] \\
a1 &= 1 \times 3 \text{ string} \\
& \text{"White" "Black" "Brown"} \\
>> a2 &= [\text{"Flower" "Vase" "Table"}] \\
a2 &= 1 \times 3 \text{ string} \\
& \text{"Flower" "Vase" "Table"} \\
>> a &= a1 + a2 \\
a &= 1 \times 3 \text{ string} \\
& \text{"White Flower" "Black Vase" "Brown Table"}
\end{aligned}
$$

Example 12. Plot 3-D Helix.

Solution: Define t as a vector of values between 0 and 10π. Define st and ct as vectors of sine and cosine values. Then plot st, ct, and t as follows-

$$
\begin{aligned}
&>> t = \theta; pi/50:10{*}pi; \\
&>> st = \sin(t); \\
&>> ct = \cos(t); \\
&>> \text{plot3}\,(st, ct, t)
\end{aligned}
$$

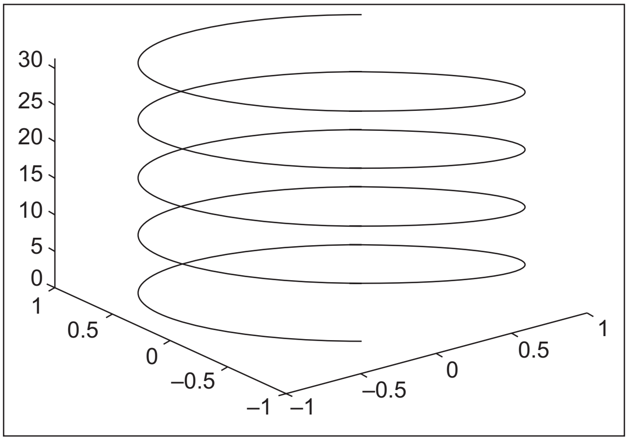


Fig. 5.9: (3-D Helix)
(*source:www.mathworks.com*)

5.4.3 Advantages of MATLAB

- It can be used in multiple ways. Like- from processing still images and creating simulation videos, to calculating, to programing and so on.
- It has predefined function which makes it easy to use.
- It is supported on many different platforms.
- It has outstanding numerous tools for visualizing technical aspects.
- It has tools which allows the user to interactively design a Graphical User Interface (GUI) for user's program.
- It's memory management is automatic.
- It has ability to call external libraries.
- In MATLAB dimension statements, pointers are not required.
- It has good online tutorials.

5.4.4 Disadvantages of MATLAB

- It is meant for technical and mathematical computing. So, it is not applicable for other fields.
- It is an interpreted language and so at times may execute more slowly than complied language. But it can be checked through proper structuring of program.
- For large computing it requires fast computer with adequate memory.

5.4.5 A Few Keyboard Shortcuts for MATLAB

Action	Keyboard Shortcut
Move to the next visible panel.	Ctrl + Tab
Move to the previous visible panel.	Ctrl + Shift + Tab
Move to the next tab in a panel.	Ctrl + Page Down
Move to the previous tab in a panel.	Ctrl + Page Up

Action	Keyboard Shortcut
Make an open tool the active tool.	• Command Window: Ctrl+0 • Command History: Ctrl+1 • Current Folder: Ctrl+2 • Workspace: Ctrl+3 • Profiler: Ctrl+4 • Figure Palette: Ctrl+6 • Plot Browser: Ctrl+7 • Property Editor: Ctrl+8 • Editor: Ctrl+Shift+0 • Figures: Ctrl+Shift+1 • Web browser: Ctrl+Shift+2 • Variables Editor: Ctrl+Shift+3 • Comparison Tool: Ctrl+Shift+4 • Help browser: Ctrl+Shift+5 On macOS systems, use the Command key instead of the Ctrl key.
Cancel the current action.	Esc (escape) For example, if you click the name of the Edit menu, the whole menu appears. Pressing Esc hides the menu again. In the Function Browser, pressing Esc up to three times has the following effects: 1. Dismiss the search history. 2. Clear the search field. 3. Close the Function Browser.

(*source:www.mathworks.com*)

Video Resource References [Source: NCERT]

Differential Equations: General and Particular Solution

Get started with MATLAB .Retrieved from

Live editor. Retrieved from

Data Analysis. Retrieved from

MATLAB Graphics

Algorithm development

App Designer.

MATLAB with other languages

Parallel Computing

Desktop and web Deployment

MATLAB in the cloud

Application of Differential Equations and MATLAB

1. Sometimes, even in diverse problems of distinct scientific fields identical differential equations are obtained. This explains us the unifying principle behind diverse phenomena. Like-propagation of light and sound in the atmosphere and of waves on the surface of a pond-all can be described by the same 2nd order partial differential equation.
2. Radioactive decay modelling, population growth, prey-predator models etc. uses concept of differential equations.
3. Differential equation are used in physics, computer graphics and vision, gaming features, robotics, to predict change in investment return over time, bank interest, flow problems, seismic waves; in the field of medicine like for modelling cancer growth or the spread of disease etc.
4. Differential equations are also used with Newton's second law of motion and the law of cooling related to the temperature of objects and its surrounding.
5. MATLAB has many applications in industry as well as in academia. It is used in the software components of washing machines, printer, automobiles, industrial machines etc. With the push of one button, MATLAB generates code and runs the hardware.

6. MATLAB is such that with just a few simple lines of coding, a user can build models without having to be an expert.
7. Technical professionals use MATLAB to study big data to gain insights.
8. Apart from above, MATLAB has many applications in wide area like-robotics, computational biology, computational finance, mechatronics etc.

CASE STUDY

A village has a population of 1000 people. The government launches a scheme to make all the villagers computer literate. For this, one villager is chosen to be the leading person, by making him/her computer literate. If computer literacy spread is proportional to the product of the number of computer-literate villagers and remaining villagers, and there are 100 computer-literate people after 10 days- then, answer the following questions:

Q.1. If $c(t)$ denotes the number of computer literate students at any time then, maximum and minimum value of $c(t)$ respectively is

(*a*) 50 and 2 (*b*) 100 and 1 (*c*) 1000 and 1 (*d*) 0

Q.2. The value of $c(10)$ is

(*a*) 50 (*b*) 100 (*c*) 1000 (*d*) 10

Q.3. Can you make a mathematical model related to computer literacy of the population, based on the given information? If yes, then get it verified by your teacher.

Check Out!!!!

Download the free trial version of MATLAB (Source: MathWorks) Now that you are aware of MATLAB, checkout whether you can get the solution of the following differential equation using a toolbox from MATLAB: $\frac{dy}{dx} = \frac{y}{x}$. Verify your answer manually!

Math Work

SUMMARY

1. An equation which contains the derivatives of the dependent variables with respect to independent variables is known as a ***differential equation***.
2. ***Order*** of a differential equation is defined as the order of the highest derivative occurring in the equation.
3. ***Degree*** of a differential equation is defined as the power of the highest order derivative occurring in the polynomial equation of derivatives after the equation is made free from radical signs and fractions.
4. Order and degree of a differential equation are always positive integers.

5. **Variable separation method:** If a differential equation can be written in the form $f(x)dx = g(y)dy$. Then we say that variables are separable and solution is $\int f(x)dx = \int g(y)dy + c$, where c is the constant of integration.
6. MATLAB stands for MATrix LABoratory.
7. MATLAB is a high level multi-paradigm language meant for technical and mathematical computing.
8. It was created by ***Cleve Moler***, in the 1970's.
9. In 1980's MATLAB was released as a commercial product.
10. **MATLAB Desktop:** It is the main place where we work. It consists of five sub windows- Command Window ,Current Directory Pane, Details (File) Pane ,Workspace Pane, Command History Pane.

Exercises

Subjective Questions

Q.1. Explain order and degree of a differential equation. Given examples.

Q.2. Solve: $(1 - x)dy - (1 + y)dx = 0$ [**Ans.** $y - x - xy = c$]

Q.3. Find the general solution of the differential equation

$$x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0$$ [**Ans.** $\sqrt{1+x^2} + \sqrt{1+y^2} = c$]

Q.4. Find the particular solution of the differential equation

$$\frac{dy}{dx} = e^{x-y} + x^2e^{-y} \text{ when } x = 0 \text{ and } y = 1$$ $\left[\textbf{Ans. } e^y = e^x + \frac{x^3}{3} + e - 1\right]$

Q.5. Explain the basics of MATLAB.

Q.6. What is MATLAB. Write its merits and demerits.

Objective Questions

Q.1. The function which satisfies the differential equation: $x^3dx + (y + 1)^2dy = 0$ is

(*a*) $y = x + c$ (*b*) $4x^4 + 5(y + 1)^2 = c$
(*c*) $3x^4 + 4(y + 1)^3 = c$ (d) $3x^2 + 4y^2 + c^2 = 0$ [**Ans.** (*c*)]

Q.2. Particular solution of $(1 + x^3)dy - x^2ydx = 0$ when $x = 1$ and $y = 2$ is

(*a*) 2 (*b*) $y^2 = 4(1 + x^2)$ (*c*) 3 (*d*) $y^3 = 4(1 + x^3)$
[**Ans.** (*d*)]

Q.3. Order of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{d^3y}{dx^3}\right)^{1/3} = 0$ is

(*a*) 2 (*b*) 3 (*c*) 1/3 (*d*) 0 [**Ans.** (*b*)]

Q.4. Degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} + y = \sin x$ is

(*a*) 2 (*b*) 3 (*c*) 1 (*d*) 4 [**Ans.** (*d*)]

Q.5. The equation $\dfrac{dy}{dx} = \dfrac{(\log y)^2}{(\log x)^2}$ is in variables separable form

(*a*) True (*b*) False [**Ans.** (*a*)]

Q.6. Abbreviation MATLAB stand for ________. [**Ans.** MATrix LABoratory]

Q.7. Match the following

Files of MATLAB		**EXTENSION**	
1.	P-files	(*a*)	.m
2.	Fig-files	(*b*)	.mat
3.	M-files	(*c*)	.fig
4.	Mat-files	(*d*)	. p

[**Ans.** 1-(*d*); 2-(*c*); 3-(*a*); 4-(*b*)]

Q.8. ________ of MATLAB are a collection of numerous functions built-in it. [**Ans.** Toolboxes]

Q.9. Basic building block of MATLAB is ________. [**Ans.** matrix]

Q.10. MATLAB was created by

(*a*) Murray (*b*) Bill Gates (*c*) Cleve Moler (*d*) Microsoft

[**Ans.** (*c*)]

Mini Project

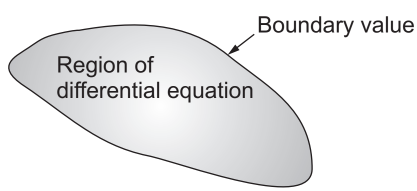
Form a group of five students each. Select a MATLAB toolbox for your group. Using this toolbox make a presentation explaining at least 10 features of that toolbox along with its examples.

Activity

Select any prey-predator model from a mathematical modelling book of your department or from online sources. Explain the importance of differential equation in that model. Give a verbal presentation of the same to your teacher!

Know More

- Other methods apart from variable-separable for solving first order first degree differential equations are for-
  (*a*) Homogeneous equations.
  (*b*) Leibnitz' Linear Equations.
  (*c*) Bernoulli's equations.
  (*d*) Exact differential equations.
  (*e*) Converting non-exact to exact differential equations.
- A boundary value problem of a differential equation consists of a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution of the differential equation which also satisfies the boundary conditions.



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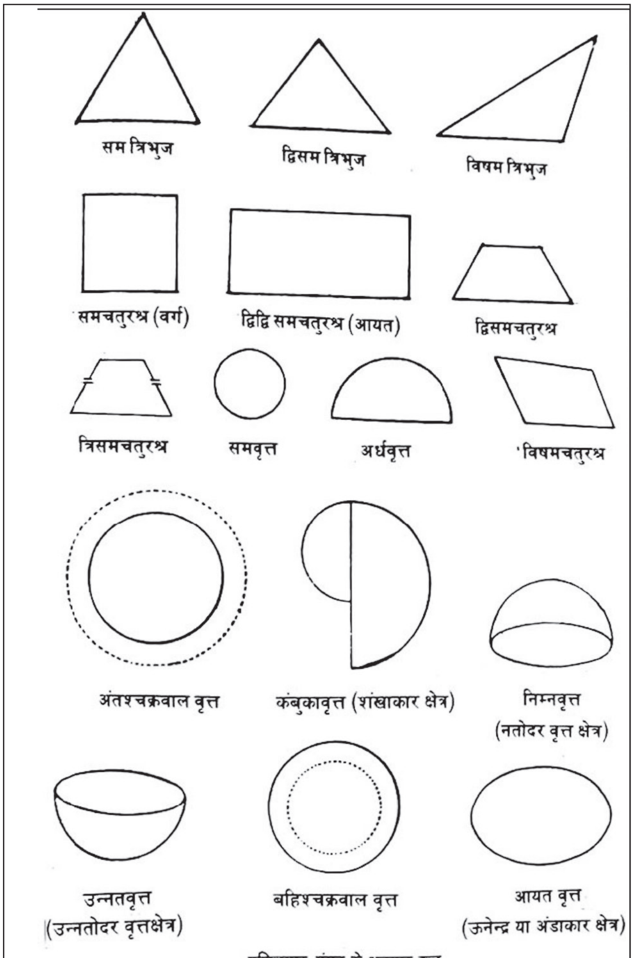


गणितसार-संग्रह के अनुसार कुछ ज्यामितीय क्षेत्रों की आकृतियां

An excerpt from *"Ganitsaar-Sangraha"* written by Indian Mathematician *Mahaveeracharya*.
(**Source:** Muley Gunakar (1992), Sansaar ke Mahan Ganitagya, Raajkamal Prakashan)

Appendices

APPENDIX-A

Mathematics Laboratory

To instigate mathematical awareness among students, mathematics laboratory gives an excellent opportunity to students in constructing knowledge by themselves under the proper guidance of the teacher.

A few activities are given below to facilitate students in comprehending mathematics from theory to abstract. With a mathematical laboratory, students learn and enjoy the principles of mathematics in a better way.

For management and maintenance of mathematics laboratory separate arrangements are desirable. But even if these are not as per norms,simple activities can be done easily in the classroom itself.

EXPERIMENT 1:

Aim

Constructing various types of conic sections.

Requirements

Transparent sheet, scissors, hardboard, adhesive, black paper.

Procedure

1. Take hardboard of convenient size and paste a black paper on it.
2. Cut a transparent sheet in the shape of sector of a circle and fold it to obtain a right circular cone as shown in Fig. a.1

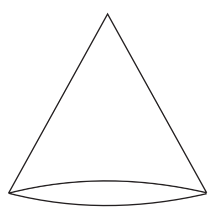


Fig. a.1

3. Form 4 more such cones of the same size using transparent sheet. Put these cones on hardboard.
4. Cut these cones with a transparent plane sheet in different positions accordingly as shown in Fig. a.2, a.3, a.4, a.5.

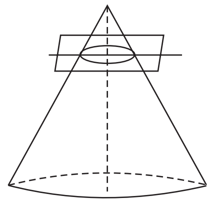


Fig. a.2

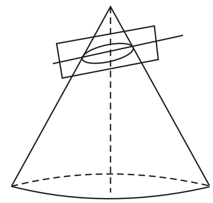


Fig. a.3

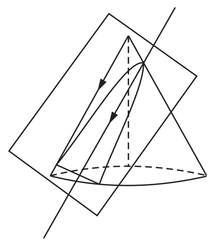


Fig. a.4

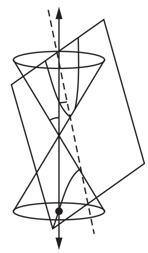


Fig. a.5

Observations

1. In fig. a.2, the transparent plane sheet cuts the cone in a manner that the sheet is parallel to the base of the cone. The conic section so obtained is a circle.
2. In fig. a.3, the plane sheet is inclined slightly to the axes of the cone, so the conic section obtained is an ellipse.
3. In fig. a.4, the plane sheet is parallel to the generator of the cone, hence the conic section so obtained is a parabola.
4. In fig. a.5, the plane is parallel to the axis of the cone, hence the sections so obtained are part of a hyperbola.

Results

1. In fig. a.2, the transparent plane sheet is _______ to the base of the cone and the section obtained is _______.
2. In fig. a.3, the plane sheet is inclined to _______ and the conic section obtained is _______.
3. In fig. a.4, the plane sheet is parallel to the _______ and the conic section so obtained is _______.
4. In fig. a.5, the plane sheet is _______ to the axis and the conic section so obtained is a part of _______.

Inferences

This activity aids in understanding various types of conic sections which have extensive applications in real life conditions and contemporary sciences.

EXPERIMENT 2:

Aim

Constructing an ellipse with given major and minor axes.

Requirements

A hardboard, black paper, thread, glue, chart paper.

Procedure

1. Take a rectangular sheet of a hardboard of a suitable size and paste a black paper on it.
2. Mark a point O on it. Draw two concentric circles with centre O and radii as given semi-major and semi-minor axis of the ellipse. Mark one of the diameters of bigger circle as AOA′ and call it a horizontal line.

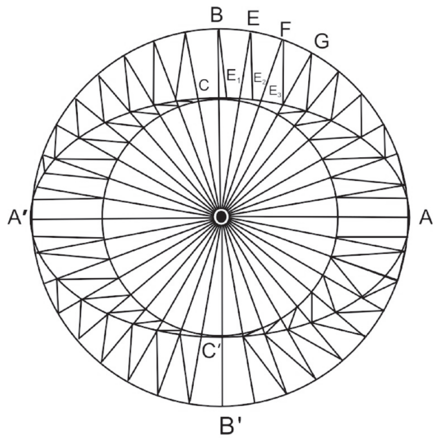


Fig. b.1

3. Draw radii of the circles in such a way that the angle between two consecutive radii is the same, say 15°.
4. Take any radius OB of the bigger circle cutting the smaller circle at C. Draw a horizontal line through C and draw a perpendicular (vertical line) from B to this horizontal line and obtain point E_1. (see Fig. b.1)
5. Repeat this process for all the radii OE, OF and so on of the bigger circle and obtain the points E_2, E_3, ... and so on.
6. Fix the nails at the points E_1, E_2, E_3, ... and join the feet of the nails by a thread and obtain a curve. (Fig. b.1)

Observations

1. The curve so obtained is an ellipse.
2. Major axis of the ellipse is AOA′ and the minor axis of the ellipse is BOB′, where BOB′ is the diameter of the smaller circle perpendicular to diameter AOA′.

Results

(*i*) OA = ________ OB = ________

(*ii*) OC = ________ OC′ = ________

(*iii*) Major axis of the ellipse ________. Minor axis of the ellipse = ________

(*iv*) Points E_1, E_2 ________ lie on ________.

Inferences

This activity may be used in making elliptical designs using thread, nail and also in elucidation of concepts such as major and minor axis of an ellipse.

Appendix-B: Assessments Aligned to Bloom's Level

Suggested Table of Specification for Question Paper Design

Unit No.	Unit Title	Teaching Hours	Distribution of Theory Marks			
			R Level	U Level	A Level	Total Marks
1.	Determinants & Matrices	11	5	6	5	16
2.	Integral Calculus	9	5	4	5	14
3.	Co-ordinate Geometry	10	4	5	5	14
4.	Vector Algebra	8	4	4	4	12
5.	Differential Equations	10	4	5	5	14
		48 hrs.	22	24	24	70

Legends; R=Remember; U= understand- A=Apply and above (Bloom's Taxonomy)

Note: This specification table provides general guidelines to assist student for their learning and to teachers to teach and assess students with respect to attainment of UO's. The actual distribution of marks at different taxonomy levels (of R, U and A) in the question paper may vary from above table.

Annexures

ANNEXURE-1

LOGARITHM

Definition

If a, x, y are any three numbers, such that $a^x = y$, then x is known as the logarithm of y to the base a. Hence, logarithm of a number to some base is the exponent by which the base must be raised in order to get that number. We can write the relation $a^x = y$ in logarithmic form as

$$\log_a y = x, \text{ hence } a^x = y \log_a y = x$$

Note:

1. Every positive real number y can be expressed in exponential form as $a^x = y$ were 'a' is also a positive real number different than unity and is called the base and 'x' is called an exponent.
2. Limitations of logarithm - $\log_a y$ is defined only when
   (i) $y > 0$ (ii) $a > 0$ (iii) $a \neq 1$
3. Logarithm of zero does not exist.
4. For a give value of y, $\log_a y$ will give a unique value.
5. $\log_a 1 = 0$.
6. $\log_y y = 1$.
7. $\log y = -1 = \log_y \dfrac{1}{y}$
8. $y = a^{\log_a y}$

Properties of Logarithms

If m, n are arbitrary positive numbers where $a > 0$, $a \neq 1$, then following are some salient properties of logarithms-

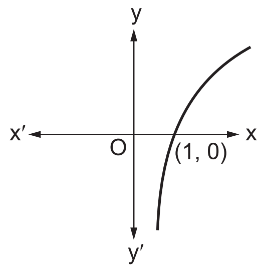
1. Fundamental logarithm's identity-
   $$a^{\log_a m} = m$$
2. $\log_a mn = \log_a m + \log_a n$
3. $\log_a \dfrac{m}{n} = \log_a m - \log_a n$
4. Base changing theorem–

   It states that quotient of the logarithm of two numbers is independent of their common base.

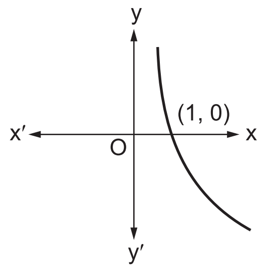
   Symbolically, $\log_b m = \dfrac{\log_a m}{\log_a b}$ where $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$

GRAPH OF LOGARITHMIC FUNCTION

$$y = \log_a x$$



where a > 1



where 0 < a < 1

CHARACTERISTIC AND MANTISSA

For any given number y, logarithm can be expressed as $\log_a y$ = Integer + Fraction. The integer part is called characteristic and the fractional part is called mantissa. When the value of $\log n$ is given, then to find digits of 'n' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $n \geq 1$) or the number of zeros after decimal & before first non-zero digit in the number (if $0 < n < 1$).

Note:

(*i*) The mantissa part of logarithm of a number is always positive ($0 \leq m < 1$)

(*ii*) If the characteristic of $\log_{10} y$ be n, then the number of digits in y is $(n + 1)$

(*iii*) If the characteristic of $\log_{10} y$ be $(-n)$, then there exist $(n - 1)$ zeros after decimal in y.

Antilogarithm

The positive real number 'n' is called the antilogarithm of a number 'm' if $\log n = m$

Thus, $\log n = m \Leftrightarrow n = \text{antilog } m$.

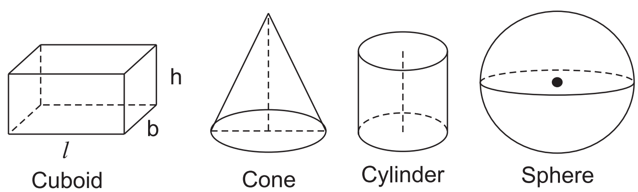
ANNEXURE-2

Solid

Requires three dimensions to describe

(*a*) Surface of solids - plane areas bounding the solid e.g. six rectangle faces bounding brick. Surface area is measured in square units.

(*b*) Volume of solids - Space occupied by a solid and is measured in cubic units.



Cuboid

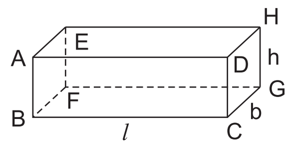
Rectangular shaped solid also known as rectangular parallelopiped (e.g. match box, brick)

(*a*) Have six rectangular faces with opposite faces parallel and congruent.

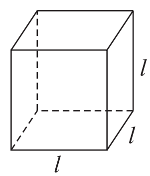
(*b*) Have twelve edges (Edge - The line segment where two adjacent faces meets).

(*c*) Three adjacent faces meet at a point called vertex and cuboid has eight vertices

(*d*) Surface area: A = $2[l \times b + b \times h + h \times l]$ square unit.



(*e*) Volume: $V = l \times b \times h$ cubic unit



Cube

Special case of cuboid having all sides equal.

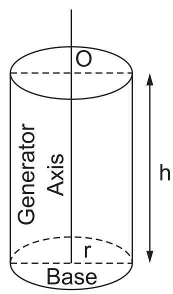
Area = $6l^2$; volume = l^3 unit cube: side $l = 1$

Volume is 1 cubic unit

Cylinder

Having lateral (curved) surface and two congruent circular cross section.

(e.g. Jar, Circular Pillars, Drums, Pipes etc.)



(*a*) Axis - Line joining the centers of two circular cross sections.

(*b*) Right circular cylinder - When an axis is perpendicular to circular cross section.

(*c*) Generators - Lines parallel to axis and lying on the lateral surface.

(*d*) Base - with cylinder in vertical position, the lower circular end is base.

(*e*) Height (*h*) - Distance between two circular faces.

(*f*) Radius (*r*) - Radius of base or top circle.

(*g*) Total surface area - Base area + curved surface area

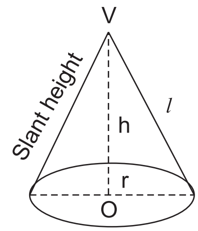
$$2r^2 + 2rh = 2r(h + r) \text{ (including two circular ends).}$$

without circular ends (Hollow cylinder) = 2

(*h*) Volume - $V = \pi r^2 h$

Cone

Has a curved surface with a vertex (V) and circular base radius: r and center (O)



Cone

(*a*) Axis - Line joining vertex and center of base circle (VO)

(*b*) Height of cone (h) - Length of VO.

(*c*) Slant height (l) - distance of vertex from any point of base circle.

$$l = \sqrt{r^2 + h^2}$$

(*d*) Right circular cone - When axis is perpendicular to base.

(*e*) The cross section of a cone parallel to base is a circle and perpendicular to base is an isosceles triangle.

(*f*) Volume - $(1/3)\pi r^2 h$ (volume of a cone is 1/3rd of volume of a cylinder with same height and base radius).

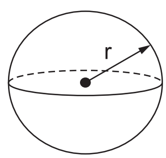
(*g*) Curved surface Area: πrl

(*h*) Total surface Area - $\pi rl + \pi r^2 = \pi r(l + r)$

(*i*) A right circular cone can be generated by rotating a right angled triangle about its right angle forming side.

Sphere

All points on its surface are equidistant from its center, the distance is called radius (r) and any line passing through center with end points on surface is called diameter.

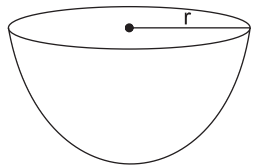


(*a*) Volume V = $(4/3)\pi r^3$

(*b*) Surface area A= $4\pi r^2$

Hemisphere

A sphere is divided into two hemi-spheres by a plane passing through center.



(*a*) Volume V = $(2/3)\pi r^3$

(*b*) Curved surface area S = $2\pi r^2$

(*c*) Total surface area A = $2\pi r^2 + \pi r^2 = 3\pi r^2$

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